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Exercise Booklet



LIMO 2018

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Tips and tricks

During the competition the following rules apply:

• Write your solutions for each problem on a separate sheet, and write down your team name, your team number and the problem number on every sheet.

• Facilities such as books, graphing calculators, mobile phones and laptops are not allowed. Non-graphing calculators may be used. You may only communicate with teammates and with the organization.

• Water is provided during the competition. For any questions, there will be a member of the organization present in the competition room.

Tips that may help you during the competition:

• Notation. The notation and/or the terminology will be explained for certain problems. Furthermore, with the natural numbers the collection {1, 2, 3, . . .} is meant, which we denote with $\mathbb{N}$.

• Order of difficulty. The problems are sorted in increasing difficulty, with easiest first. Each problem is worth 10 points.

• Read the problems carefully. If you start solving too early, you may have overlooked important information. Sometimes a problem contains a (hidden) hint that indicates what strategy to use.

• Be a team. Divide the tasks, but ask each other for help if needed. Consider where everyone's qualities lie, and check each other's work during the match; maybe there are still mistakes.

• Try to collect at least some points. A partial solution may still earn you points. If you do not have the solution for a specific sub-problem, you may still use the result for all subsequent sub-problems.

• Find a pattern. For example, if you have to prove something for all $n \in \mathbb{N}$, tro to see what happens for n = 1 or n = 2. Discover a pattern and prove that this pattern continues.

• "Houd het gezellig" is Dutch for enjoy yourself and be nice to others.

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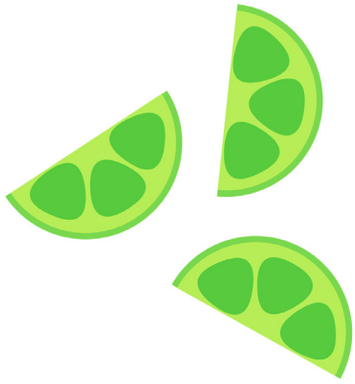

This Exercise Booklet is a product of the LIMO-committee 2018:
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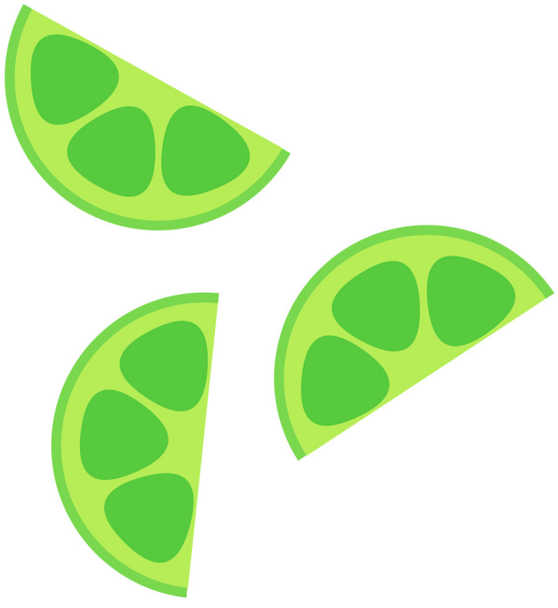
1. Even triangles are odd

Harry Smit MSc., Universiteit Utrecht

Suppose you have a triangle that is subdivided into other triangles, i.e. this triangle is completely tiled with smaller triangles, without any overlap. Furthermore, assume that no three vertices (not even those of different triangles) lie on a straight line.

(a) Let k be an odd positive integer. Prove that it is possible to make a subdivision into exactly k triangles.

(b) Let k be an even positive integer. Prove that it is impossible to make a subdivision into exactly k triangles.



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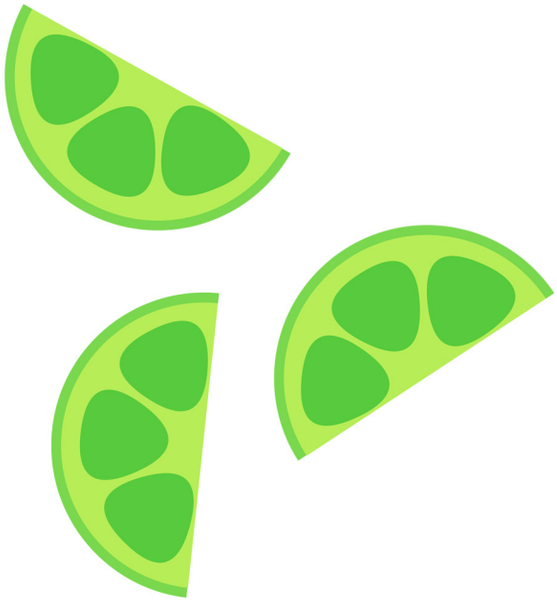
2. Self-divisible numbers

Ir. Harold de Boer, Transtrend BV

We define a positive integer to be self-divisible when each digit of the number (starting from the second digit) is a divisor of the number formed by the digits preceding it. For example, 42771 is self-divisible because 4 is a multiple of 2, 42 is a multiple of 7, 427 is a multiple of 7 and 4277 is a multiple of 1.
We define G_N as the set of self-divisible numbers that consist of N different digits. So, 42771 is an element of the set G_4.

(a) Determine the largest N for which the set G_N is not empty.

(b) Determine the smallest number in this particular set G_N.

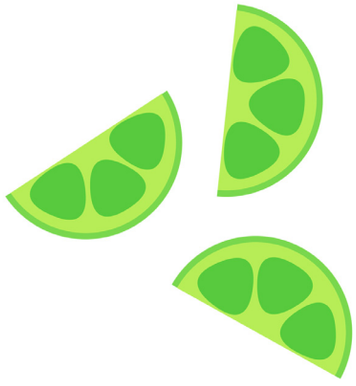


3. Ramifixation

Prof. dr. Hendrik Lenstra, Universiteit Leiden

A *tree* is a finite connected graph without cycles. An *automorphism* of a graph is a map σ that permutes both the set of vertices and the set of edges of the graph, in such a way that an edge e connects two vertices v and w if and only if $\sigma(e)$ connects $\sigma(v)$ and $\sigma(w)$.

Prove that every tree has a vertex or an edge that is mapped to itself by every automorphism of the tree.



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4. A Kurt Mahler[1] style power series

Prof. dr. Gunther Cornelissen, Universiteit Utrecht

Fix a prime number p. If an integer n is written as $n = p^r u$ where u is coprime to p, we define $|n|_p := p^{-r}$. Consider the power series

$$f(z) := \sum_{n \geq 1} |n|_p z^n.$$

(a) Show that the power series defining $f(z)$ converges for all complex numbers z with $|z| < 1$.

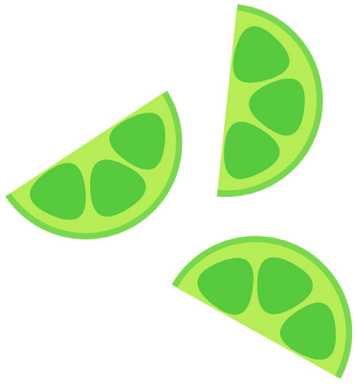
(b) Prove that for $|z| < 1$,

$$f(z) = \frac{z}{1-z} - \frac{z^p}{1-z^p} + \frac{1}{p} f(z^p).$$

(c) Prove that the series defining $f(z)$ diverges at a dense set of points in the complex unit circle by showing that

(i) it diverges at all p-power roots of unity, i.e., at all $z \in \mu_{p^\infty} := \{e^{2\pi ki/p^r} : k, r \in \mathbf{Z}\}$;

(ii) μ_{p^∞} is dense in the unit circle (i.e, for every $z_0 \in \{|z| = 1\}$ and $\varepsilon > 0$ there exists $\zeta \in \mu_{p^\infty}$ with $d(z_0, \zeta) < \varepsilon$ where d is distance along the circle).



[1]The famous number theorist Kurt Mahler (1903-1988) studied these kinds of problems in the 1930's. Between 1934 and 1936 he worked - after from escaping Germany via Manchester - in Groningen, where he had a bicycle accident, after which he walked with a limp for the rest of his life.

5. Chebyshev polynomials

Prof. dr. Jaap Top, Rijksuniversiteit Groningen

The wikipedia page describing 'Chebyshev polynomials' briefly mentions, besides the usual Chebyshev polynomials of the first kind T_n and the ones of the second kind U_n also a sequence of polynomials C_n. The sets of polynomials are related by $C_n(x) = 2T_n(\frac{x}{2})$ and $T_n' = nU_{n-1}$. This problem defines and asks for some properties of the polynomials C_n.

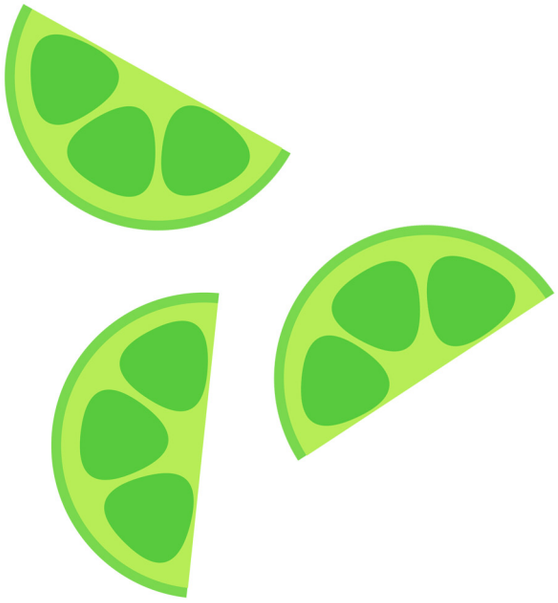
(a) Show that for any integer $n \geq 1$ a *unique* real polynomial C_n in one variable exists such that

$$x^n + x^{-n} = C_n(x + x^{-1})$$

(here x is also a variable).

(b) Show that $(4 - t^2)C_n''(t) - tC_n'(t) + n^2C_n(t) = 0$.

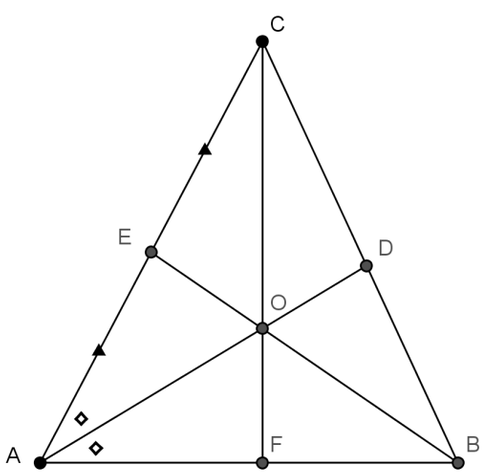
(c) Show that for every $n \geq 2$ the polynomial C_n mod 2 (which has coefficients in $\mathbb{Z}/2\mathbb{Z}$) can be factored as a product $f^2 \cdot g$ for polynomials f, g with coefficients in $\mathbb{Z}/2\mathbb{Z}$ with moreover f of degree at least 1.

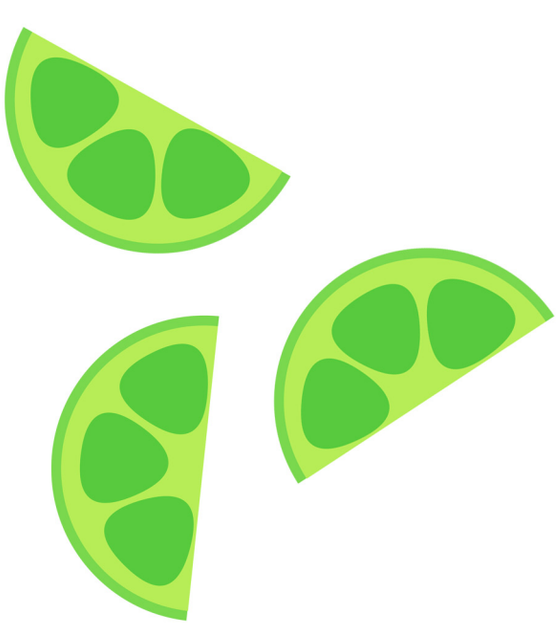


6. Medium competition in bisecting the altitude

Eduardo Ruíz Duarte MSc., Rijksuniversiteit Groningen

Find an example of a non-equilateral triangle ABC with integer sides for which the altitude from A, the bisector at B, and the median at C are concurrent, that is, they intersect at a single point. Hint: it is possible to find such a triangle with a perimeter no greater than 50.





7. Mathematicians like to play with colors

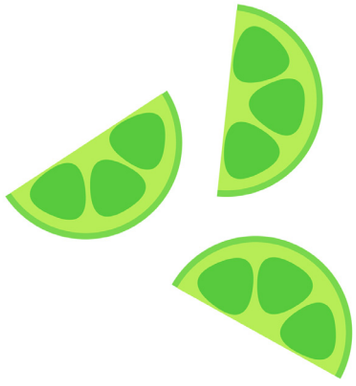
Stijn Cambie MSc., Radboud Universiteit Nijmegen

A *finite coloring* of the nonnegative integers is a map $f\colon \mathbb{N} \mapsto S : n \mapsto f(n)$, where S is a finite set. Given a coloring f, a set X is called *monochromatic* (MC) when every element is colored in the same color, i.e. for all $x_1, x_2 \in X$ one has $f(x_1) = f(x_2)$.
A theorem of Schur says that any finite coloring of $\mathbb{N}$ contains a MC set $\{x, y, x+y\}$, where $x, y \in \mathbb{N}$.

(a) Prove that any finite coloring of $\mathbb{N}$ contains a MC set $\{a, b, ab\}$ as well, where $a, b > 1$.

(b) Given a coloring $f\colon \mathbb{N} \mapsto \{B, W\}$. Does there exist a MC set of the form $\{a, b, ab+a\}$?

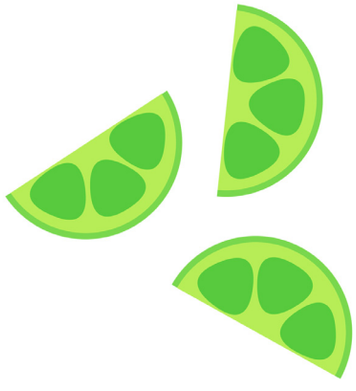
(c) Given a finite coloring of $\mathbb{N}$. Does there exists a MC set of the form $\{a, b, ab, a(b{+}2018)\}$?



8. Typing a's and b's

Dr. Daniel Valesin, Rijksuniversiteit Groningen

I type letters at random in my keyboard. Each letter I type is equally likely to be any of the 26 (lowercase) letters of the alphabet, chosen independently of what has been typed before. I stop typing when I type the letter `a` three times in succession (that is, with no other letter in between). What is the expected number of times I type the letter `b`?



9. Matrix-valued function in one variable

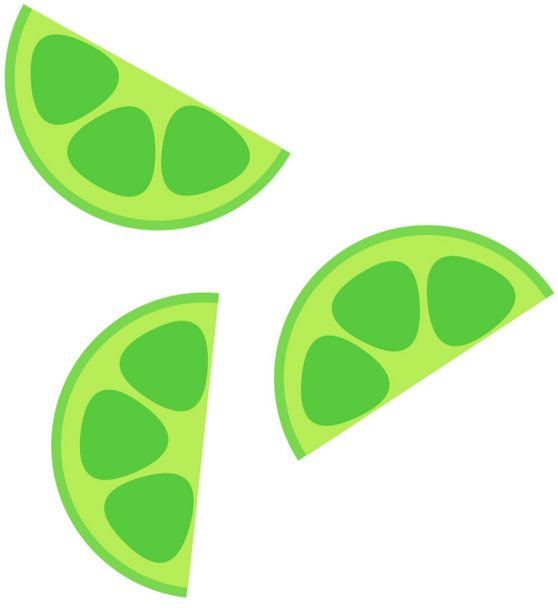
Leslie Molag MSc., Katholieke Universiteit Leuven

Let $f : \mathbb{R} \longrightarrow \mathbb{R}^{3\times 3}$ be a C^∞ function such that $f(0) = \mathbb{I}$ and $\det f(x)$ is constant.

(a) Prove that $f'(0)^3 = 0$ when f is a linear function, i.e. when $f(x) = \mathbb{I} + xf'(0)$.

(b) Show that it is not guaranteed that $f'(0)^3 = 0$.

(c) Is $f'(0)^3 = 0$ guaranteed under the additional requirement that $f''(0) = 0$?



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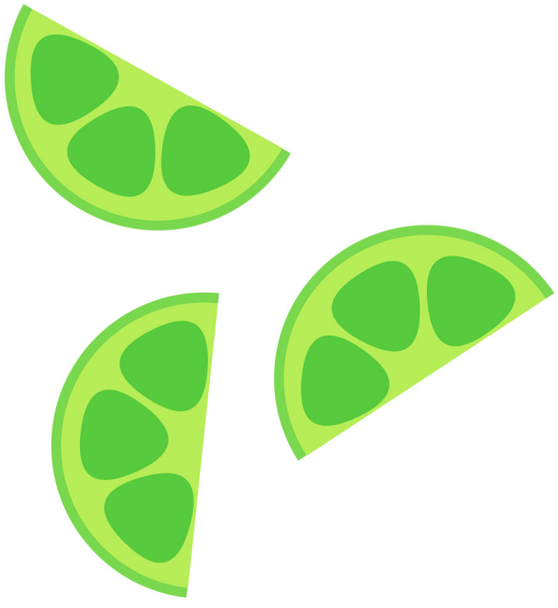
10. Rational points

Prof. dr. Frans Oort, Universiteit Utrecht

Consider a prime number $p \equiv 3 \pmod 4$. Show:

$$\#\{(x, y) \in (\mathbb{F}_p)^2 \mid 1 = x^2 + y^2 + x^2y^2\} = p + 1;$$

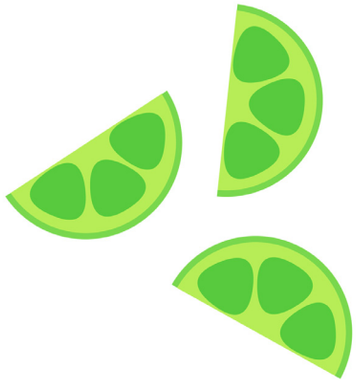
we write $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$.



11. How about this question?

Prof. dr. Lex Schrijver, Universiteit van Amsterdam

Let L be a linear subspace of $(\mathbb{Z}/2\mathbb{Z})^n$. Call an element x of L *maximal* if the support of x is not contained in the support of any other element of L. Prove that the number of maximal elements of L is odd.



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12. Restrict the extension

Prof. dr. Robert Tijdeman, Universiteit Leiden

Let $(v_n)_{n=0}^{\infty}$ be a sequence in $\mathbb{Z}^2$ with $v_0 = (1,0), v_1 = (0,1)$ and for $n = 1, 2, ...$

$$v_{n+1} = v_n + 2v_{n-1} \text{ or } v_{n+1} = v_n - 2v_{n-1}.$$

(a) Prove that for $n = 2, 3, \ldots$ the set

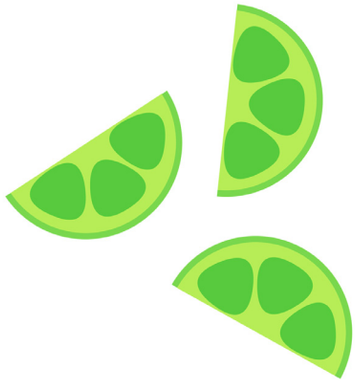
$$V_n := \left\{ \sum_{i=0}^{n-1} \varepsilon_i v_i \text{ with } \varepsilon_i \in \{0,1\} \text{ for } i = 0, 1, \ldots n-1 \right\}$$

consists of 2^n distinct points.

(b) Prove that the diameter of the convex hull of V_n for $n \geq 6$ is at least $(\sqrt{2})^n (> 1.414^n)$.

(c) Prove that the signs can be chosen in such a way that the diameter of the convex hull of V_n for $n = 2, 3, \ldots$ is at most 1.67×1.601^n.

P.S. The convex hull W of V is the smallest set such that if $v, w \in V$, then the connecting line segment belongs to W.



Exercises made by:
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Ir. Harold de Boer
Prof. dr. Hendrik Lenstra
Prof. dr. Gunther Cornelissen
Prof. dr. Jaap Top
Eduardo Ruíz Duarte MSc.
Stijn Cambie MSc.
Dr. Daniel Valesin
Leslie Molag MSc.
Prof. dr. Frans Oort
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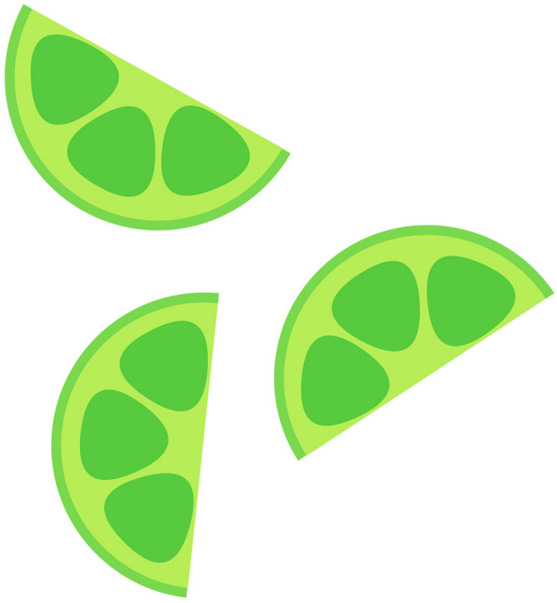

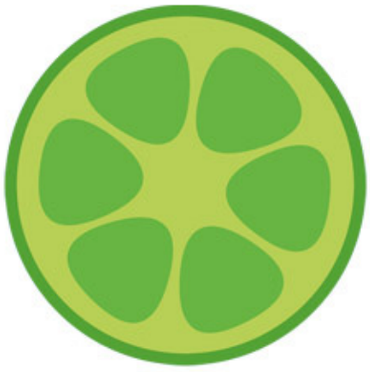


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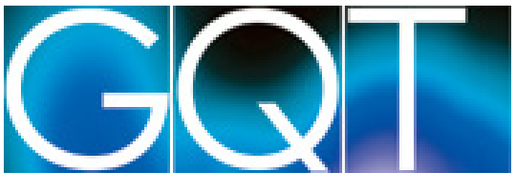
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