LIMO 2021

EXERCISE BOOKLET

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DIAMANT
Discrete, Interactive and Algorithmic Mathematics, Algebra and Number Theory
KWG
transtrend
ASML
university of groningen
Utrecht University
TU/e EINDHOVEN UNIVERSITY OF TECHNOLOGY
DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
FLOW TRADERS

This exercise booklet is a product of the LIMO-committee 2021:
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Exercises: J. Ittersum, R. van Bommel, F. van der Bult, I. Kryven, R. Versendaal, H. Smit, M. Staps, M. Daas, H. de Boer, H. Lenstra, M. Kool, S. Cambie, J. Top en D. Gijswijt.

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Rules and tips

The following **rules** apply during the competition:

- You will have until four pm to work on the assignments. Then you have until half past four to upload your elaborations to Google forms. To do this, make a scan or photo of your results and make one PDF for every exercise.
- Make each assignment on a separate sheet and provide them with a team name and assignment number. Number your pages (1/2, 2/2).
- Tools such as books and calculators are not allowed. Laptops and phones are only allowed for communication with teammates and with the organization. Definitions and propositions may not be looked up.
- If you have any questions about the competition / assignments during the competition, please send an email to limo2020@a-eskwadraat.nl.

Tips that can help you during the match:

- **Notation.** For various assignments, the notation and/or the terminology is explained in italics at the bottom. With the natural numbers we mean the set $\{1, 2, 3, ...\}$, which is denoted by $\mathbb{N}$.
- **Order of difficulty.** We have tried to sort the problems in order of difficulty. That is to say, we think that for the first exercises there is an average more points will be achieved than for the later assignments.
- **Read carefully what is stated in the assignment.** If you start too fast, important information can be overlooked. Sometimes the question contains a (hidden) hint which indicates what you could do. If you get stuck, you can also decide to read the assignment again. Also make sure to use all the information provided in the statement and especially only the information that has been given.
- **Be a team.** Divide the tasks, so that you don't do double work, and ask each other for help if you are stuck. Think about where everyone's qualities lie. Look at each other's work during the competition; often mistakes can still be corrected.
- **Gather points.** If you can't figure it out, write down what you have proven. This can be relevant for proving the statement in question and you can often get partial scores for that. In any case, results from previous years show that not all points are often scored for a problem. You can use the (non proven) result of a partial assignment, to solve the following partial assignment.
- **Don't get stuck in wrong thoughts.** It is often wise to look at a problem from a different point of view. It often helps to rewrite given terms or manipulate data. If you make little progress, you can also work at another problem and let someone else look at your problem.
- **Find a pattern.** For example, if you have to prove something for all $n \in \mathbb{N}$, try small cases: see what happens for $n = 1$ or $n = 2$. Discover a pattern and proof that this pattern continues with larger numbers.

1. Functions with certain symmetries

J.W. (Jan-Willem) M. van Ittersum, MSc.
Universiteit Utrecht

The sine function is a periodic function, i.e.,

$$\sin(2\pi(x+1)) = \sin(2\pi x),$$

and the polynomial in $\frac{1}{x}$ given by $\varphi(x) = x^{-2021} + 1$ is an example of a function satisfying

$$\varphi(x) = x^{-2021}\varphi\big(\tfrac{1}{x}\big).$$

In this exercise we are looking for functions f combining these two properties, that is,

(i) $f(x) = f(x+1)$;

(ii) $f(x) = x^{-2021} f\big(\frac{1}{x}\big)$ for all $x \neq 0$.

a) Show that there exists a *non-constant* function $f : \mathbb{R} \to \mathbb{R}$ satisfying the properties (i) en (ii) for all $x \in \Gamma$.

b) Show that there does not exists a *continous* non-constant function $f : \mathbb{R} \to \mathbb{R}$ satisfying the properties (i) en (ii) for all $x \in \Gamma$.

Let $\mathfrak{h} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. We construct a non-constant continuous periodic function $f : \mathfrak{h} \to \mathbb{C}$ which also satisfies the second property, as follows,

$$f(z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{((8m+5)z + (8n+5))^{2021}}.$$

You may use without proof that this double sum converges absolutely for $z \in \mathfrak{h}$: hence, the value of this double sum does not depend on the order of summation.

c) Show that $f(z) = f(z+8)$ and $f(z) = z^{-2021} f\big(\frac{1}{z}\big)$ for all $z \in \mathfrak{h}$.[1]

[1] Functions such as in part (c) are called *modular forms*. They play an important role in the proof of *Fermat's last theorem*, as well as in the *sphere packing problem* in dimension 8 and 24.

2. Surjective polynomials

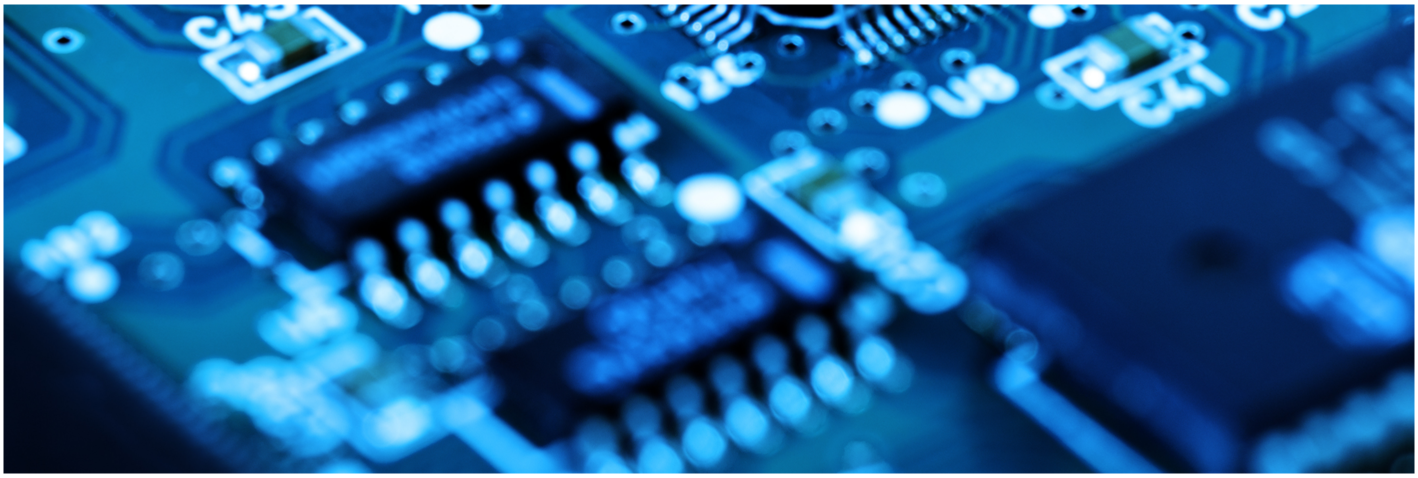
dr. R. (Raymond) van Bommel
Massachusetts Institute of Technology

a) Does there exist a $P \in \mathbb{Q}[x]$ such that the function $\mathbb{Q} \to \mathbb{Q}\colon x \mapsto P(x)$ is surjective and $\deg P > 1$?

b) For which prime numbers p does there exist a polynomial $P \in \mathbb{Z}[x]$ such that the function $\mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}\colon x \mapsto P(x) \bmod p$ is surjective and $\deg P > 1$?

c) Does there exist a polynomial $P \in \mathbb{Z}[x]$ with $\deg P > 1$, such that the function $\mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}\colon x \mapsto P(x) \bmod p$ is surjective for infinitely many prime numbers p?

d) Does there exist a polynomial $P \in \mathbb{Z}[x]$ with $\deg P > 1$, such that the function $\mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}\colon x \mapsto P(x) \bmod p$ is surjective for all prime numbers p?



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3. A quadrant of quadrilaterals

Dr. F. (Fokko) J. van de Bult
Technische Universiteit Delft

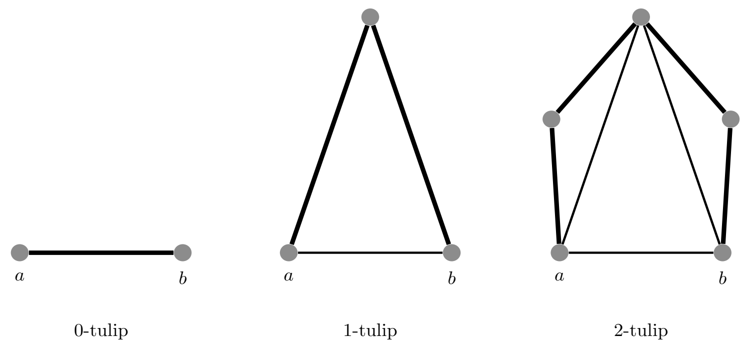
Consider a convex quadrilateral $ABCD$ without two parallel sides. Associated to this quadrilateral we define four parallellograms. These are obtained by removing one of the four vertices of the original quadrilateral and creating a parallellogram which contains three of the remaining vertices and which has two sides in common with the original quadrilateral $ABCD$.

For example, with $A = (0,0)$, $B = (1,0)$, $C = (2,1)$, and $D = (3,5)$, the the parallellogram corresponding to triangle ABC has fourth vertex $D' = (1,1)$.

Show that exactly one of these four parallellograms is contained completely within the original quadrilateral $ABCD$.

4. Fringed tulip

dr. ir. Rik Versendaal en dr. Ivan Kryven,
Universiteit Utrecht



An n-tulip is the following iterative construction: A 0-tulip consist of two initial vertices a and b connected with a link. Iteratively, an $(n+1)$-tulip is obtained from an n-tulip by glueing a triangle to each newly added link at the previous iteration.

A fringed n-tulip is an n-tulip in which each link is removed with probability $1-p$. We say that a and b are connected with a path, if there is at least one way to travel from a to b by following the links. We are interested in $f_n(p)$, the probability that there is a path from a to b in a fringed n-tulip. Note that by definition, $f_0(p) = p$, because the only possible path is the link (a, b) itself.

Show that:

a) $f_n \in C^\infty(0, 1)$ for $n \in \mathbb{N}_0$.

b) $f_n(p)$ converges for all $p \in (0, 1)$.

Let $F : (0, 1) \to [0, 1]$ be defined by $F(p) := \lim_{n\to\infty} f_n(p)$. Show that:

c) $F \in C^0(0, 1)$ and $F \notin C^1(0, 1)$.

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5. From sumrise to sumset

dr. H. J. (Harry) Smit en M. (Merlijn) Staps, MSc.
Max Planck Institute Bonn & Princeton University

Although the LIMO's problems (*sommen*) are different each year, in this problem we will consider sets whose sums (*sommen*) are all equal.
Suppose we have some red and blue cards with an integer number on each card, such that for every integer k the number of ways to select a number of red cards with sum k equals the number of ways to select a number of blue cards with sum k.

a) Prove that if the cards contain only positive numbers, then for every positive integer ℓ it holds that the number of red cards containing ℓ equals the number of blue cards containing ℓ.

b) Prove that if the cards may also contain negative integers, then for every positive integer ℓ it holds that the number of red cards containing ℓ or $-\ell$ equals the number of blue cards containing ℓ or $-\ell$.

(c) Prove that if the red and blue cards do not contain exactly the same numbers (i.e., the number of red cards containing ℓ differs from the number of blue cards containing ℓ for at least one integer ℓ), then we can select a positive number of red cards such that the sum of the numbers on these cards equals 0.

In this problem you are allowed to use the results of earlier parts, even when you have not yet solved them.

6. A pretty problem on positive polynomials

M. (Mike) Daas, MSc.
Universiteit Leiden

Let P be a polynomial with positive coefficients. Determine the maximum of $P(x)^2/P(x^2)$ over all $x \in \mathbb{R}$. For which x is this maximum attained?

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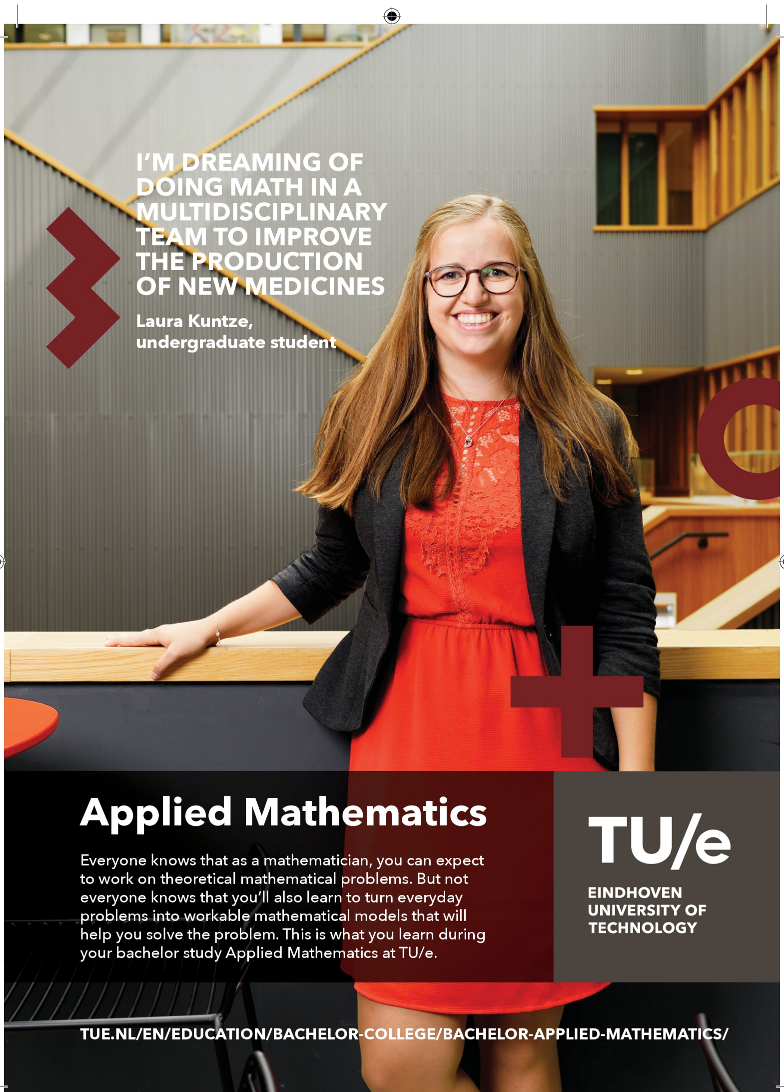


Applied Mathematics ad 170x240 mm.indd 1 31-07-20 15:42

Over het Koninklijk Wiskundig Genootschap (KWG)

Wat is het KWG? In 1778 opgericht, beoogt het KWG een verbindend orgaan te zijn voor de wiskundige beroepsgroep en een stimulans te bieden voor wiskundige activiteiten. Daarnaast vormt het KWG samen met de Nederlandse Vereniging voor Wiskundeleraren de twee pijlers van het Platform Wiskunde Nederland (PWN) dat politieke belangen van wiskundig Nederland behartigt.

Activiteiten van het KWG zijn o.a.:

- Uitbrengen van Pythagoras, een wiskundetijdschrift voor scholieren.
- Ondersteunen van de Kaleidoscoopdagen (die georganiseerd worden door de studieverenigingen).
- Organisatie van het jaarlijkse Nederlands Mathematisch Congres (voor alle wiskundigen in Nederland, i.h.b. voor wiskundigen werkzaam aan de universiteiten).
- Organisatie van het Wintersymposium (voor wiskundeleraren).
- Uitbrengen van Nieuw Archief voor de Wiskunde (4x per jaar voor alle leden; met informatie en artikelen over wiskunde voor algemeen wiskundig publiek).
- Uitbrengen van Indagationes Mathematicae (een internationaal wetenschappelijk tijdschrift).
- Beheren van Nederlandse wiskundige nalatenschap, bijv. het archief van Brouwer.

Wat kan het KWG betekenen voor wiskundestudenten?

- Één jaar gratis lidmaatschap (m.a.w.: 4x gratis het Nieuw Archief voor de Wiskunde.)
- Korting op het lidmaatschap zo lang je studeert.
- Goedkoop bijwonen Nederlands Mathematisch Congres.

Wie zit in het bestuur van het KWG? (anno najaar 2019) Danny Beckers (VU), Theo van den Bogaart (HU), Sonja Cox (UvA), Marije Elkenbracht (ABN AMRO), Barry Koren (TUE), Marie-Colette van Lieshout (CWI/UT), Michael Műger (RU), Wioletta Ruszel (UU) en Jan Wiegerinck (UvA).

De bestuursleden zijn tevens aanspreekpartners op de verschillende universiteiten.

Vragen? Kijk op de website: wiskgenoot.nl, of neem contact op met de secretaris: secretaris@wiskgenoot.nl.

7. Mirror reflex triangles

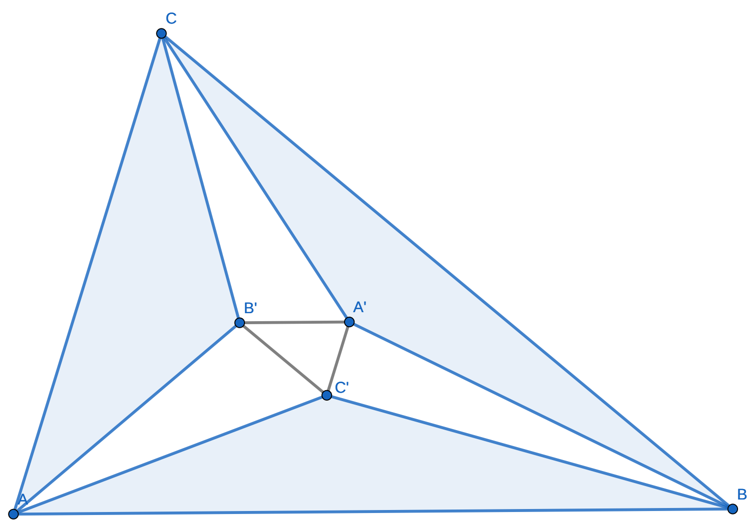
Ir. H. (Harold) de Boer
Transtrend BV

Let ABC be a triangle. Somewhere within triangle ABC there is a smaller triangle $A'B'C'$ such that:

- $A'B'$ is parallel to AB,
- $B'C'$ is parallel to BC,
- $C'A'$ is parallel to CA,
- The surface area of $A'B'C'$ is f^2-times that of ABC.

Determine the cumulative surface area of the triangles BCA', CAB' and ABC' as a fraction of the surface area of ABC, expressed in terms of f.

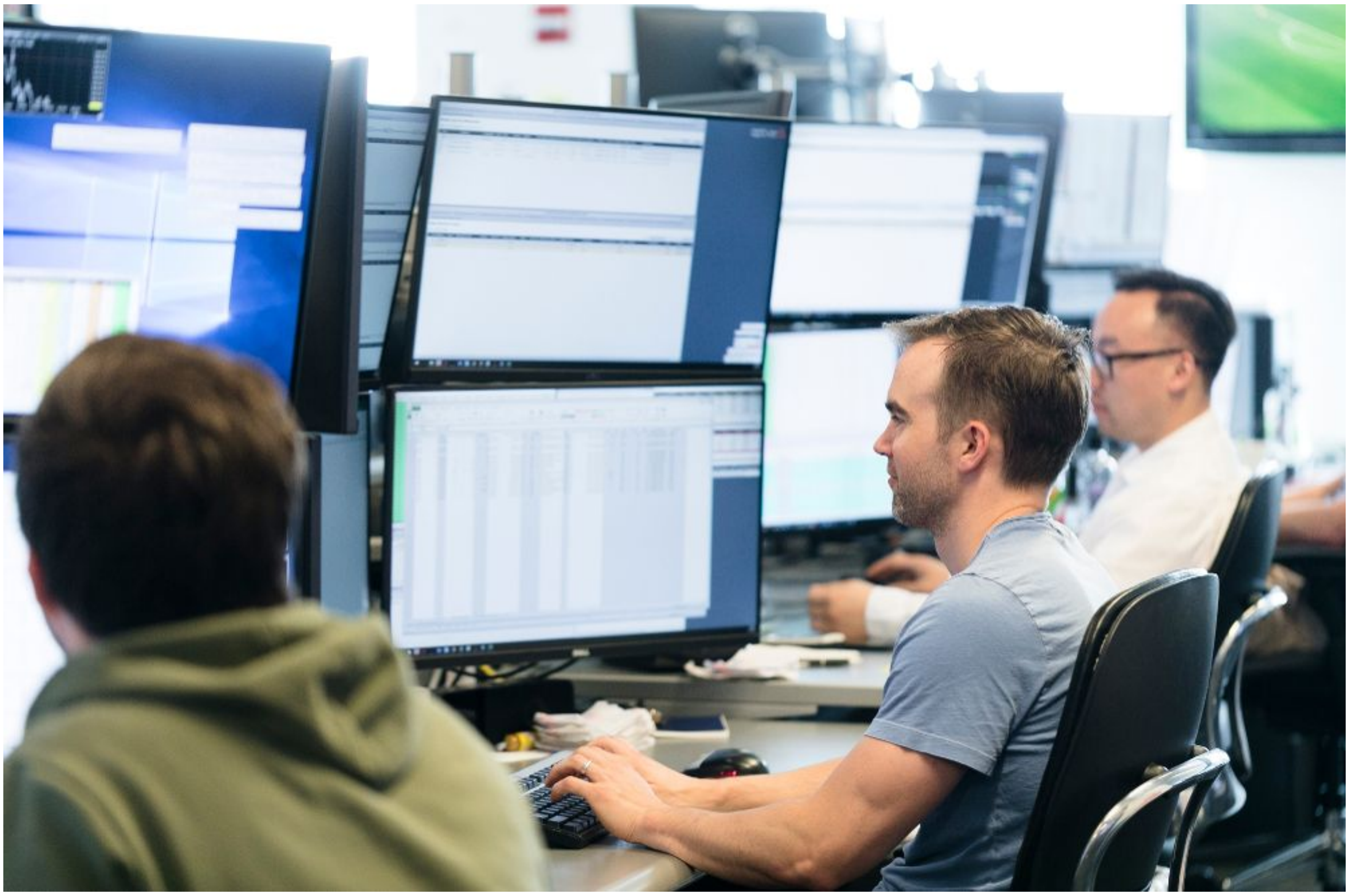
Note: Your solution should hold for any triangle ABC, irrespective of the exact location of $A'B'C'$ within ABC.



8. Sums of invertable matrices

Prof. dr. H. W. (Hendrik) Lenstra
Universiteit Leiden

Let n be a positive integer, K be a field, and A be an $n \times n$-matrix over K that is not the sum of two invertable $n \times n$-matrices over K. Prove that $n = 1, \#K = 2, A = (1)$.



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9. Superpositions of partitions

dr. M. (Martijn) Kool
Universiteit Utrecht

A *partition* is a sequence of non-negative integers $\lambda = \{\lambda_i\}_{i>0}$, such that $\lambda_i > 0$ for finitely many i and $\lambda_i \geq \lambda_{i+1}$ for all $i > 0$. We call $|\lambda| := \sum_i \lambda_i$ the *size* of λ. Define Λ to be the collection of partitions with positive size

a) Prove that

$$1 + \sum_{\lambda \in \Lambda} q^{|\lambda|} = \prod_{n=1}^{\infty} \frac{1}{1-q^n}.$$

The *support* of a partition λ is the function $f_\lambda : \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \to \{0,1\}$ with

$$f_\lambda(i,j) := \begin{cases} 1 & \text{if } j \leq \lambda_i \\ 0 & \text{else.} \end{cases}$$

A *flat partition* is a sequence of non-negative integers $\pi = \{\pi_{ij}\}_{i,j>0}$ such that $\pi_{ij} > 0$ for finitely many i, j and $\pi_{ij} \geq \pi_{i+1,j}$,$\pi_{ij} \geq \pi_{i,j+1}$ for all $i, j \geq 1$. We call $|\pi| := \sum_{i,j} \pi_{ij}$ the *size* of π. Define Π to be the collection of all flat partitions with positive size. For a flat partition $\pi \in \Pi$ we assign a weight w_π as follows. Set

$$W_\pi := \Big\{ \{n_\lambda\}_{\lambda \in \Lambda} : n_\lambda \in \mathbb{Z}_{\geq 0} \ \forall \lambda \in \Lambda \text{ en } \pi_{ij} = \sum_{\lambda \in \Lambda} n_\lambda \cdot f_\lambda(i,j) \ \forall i,j \geq 1 \Big\},$$

and define the weight of π as

$$w_\pi := \prod_{\{n_\lambda\}_{\lambda \in \Lambda} \in W_\pi} \prod_{\lambda \in \Lambda} \frac{1}{n_\lambda !}.$$

b) Prove

$$1 + \sum_{\pi \in \Pi} w_\pi p^{\pi_{11}} q^{|\pi|} = \exp\Big(p \sum_{\lambda \in \Lambda} q^{|\lambda|} \Big).$$

c) Prove

$$1 + \sum_{\pi \in \Pi} w_\pi \prod_{n=1}^{\pi_{11}} (N - (n-1)) q^{|\pi|} = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^N},$$

for all $N \in \mathbb{Z}_{>0}$.

10. Relative prime count is not relative prime

drs. S. (Stijn) Cambie
Radboud Universiteit Nijmegen

The Euler totient function ϕ is defined by mapping a positive integer n with prime factorization $\prod_{i=1}^{j} p_i^{e_i}$ to

$$\prod_{i=1}^{j}(p_i - 1)p_i^{e_i-1}.$$[2]

What is the smallest ratio m/n such that for all positive integers k, we have $\phi(k!)^n \mid (k!)^m$?

[2] This is equal to the number of integers between 0 and n realtively prime to n.

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Probleem:

Een getal is een tweemacht wanneer het kan worden geschreven als 2^k met k een geheel getal ≥ 0.

Een getal is een reekssom als het de som is van een reeks van minimaal 2 opeenvolgende positieve gehele getallen, bijvoorbeeld $15 = 4 + 5 + 6$.

Bewijs dat alle positieve gehele getallen of een tweemacht zijn, of een reekssom, maar nooit beide.

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11. Real periodic orbits

Prof. dr. J. (Jaap) Top
Rijksuniversiteit Groningen

Starting from a polynomial $p(x)$ in one variable x with $p(x)$ having real coefficients, and a complex number a_1, one defines a sequence $(a_n)_{n\geq 1}$ by iterating: so $a_2 = p(a_1)$, $a_3 = p(a_2)$ et cetera: $a_{m+1} = p(a_m)$ for every integer $m \geq 1$.

The sequence $(a_n)_{n\geq 1}$ is called a periodic orbit of the iteration, if $a_m = a_1$ for some $m > 1$. The periodic orbit is called real, if moreover $a_n \in \mathbb{R}$ for every n. As a simple example, take $p(x) = x^2$. Then every sequence starting with $a_0 = e^{2\pi ir}$ for some rational number r with an odd denominator is a periodic orbit. Also the sequence starting with $a_0 = 0$ is periodic. It turns out that these are the only periodic orbits for this polynomial, so in particular the only real periodic orbits in this case are the ones starting with $a_1 \in \{0, 1\}$. Many of periodic orbits for the given polynomial are not real.

The situation is very different for the polynomial $q(x) = 2x^2 - 1$: Show that for $q(x)$ all periodic orbits are real!

12. Colouring the line with infinitely many colours

Prof. dr. D. (Dion) Gijswijt
Technische Universiteit Delft

We seek a surjective function $f : \mathbb{R} \to \mathbb{Z}$ such that for all $a, b, c \in \mathbb{R}$ the following holds:

$$a + c = 2b \implies \#\{f(a), f(b), f(c)\} < 3.$$

Does such a function exist?