LIMO 2021

SOLUTION BOOKLET

optiver

DIAMANT
Discrete, Interactive and Algorithmic Mathematics, Algebra and Number Theory

ASML

KWG

transtrend

TU/e EINDHOVEN UNIVERSITY OF TECHNOLOGY DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE

university of groningen

FLOW TRADERS

Utrecht University

This solution booklet is a product of the LIMO-committee 2021:
Ludo Dekker, Lizanne van der Laan, Jorke de Vlas, Jasper Oostlander, Stein Meereboer, Rinske Oskamp en Mieke Wessel
e-mail: `limo2020@a-eskwadraat.nl`
website: `limo.a-eskwadraat.nl`
Exercises: J. Ittersum, R. van Bommel, F. van der Bult, I. Kryven, R. Versendaal, H. Smit, M. Staps, M. Daas, H. de Boer, H. Lenstra, M. Kool, S. Cambie, J. Top en D. Gijswijt.

Table of contents

---

1. Functions with certain symmetries

J.W. (Jan-Willem) M. van Ittersum, MSc.
Universiteit Utrecht

The sine function is a periodic function, i.e.,

$$\sin(2\pi(x+1)) = \sin(2\pi x),$$

and the polynomial in $\frac{1}{x}$ given by $\varphi(x) = x^{-2021} + 1$ is an example of a function satisfying

$$\varphi(x) = x^{-2021}\varphi\big(\tfrac{1}{x}\big).$$

In this exercise we are looking for functions f combining these two properties, that is,

(i) $f(x) = f(x+1)$;

(ii) $f(x) = x^{-2021} f\big(\frac{1}{x}\big)$ for all $x \neq 0$.

a) Show that there exists a *non-constant* function $f : \mathbb{R} \to \mathbb{R}$ satisfying the properties (i) en (ii) for all $x \in \mathbb{R}$.

b) Show that there does not exists a *continous* non-constant function $f : \mathbb{R} \to \mathbb{R}$ satisfying the properties (i) en (ii) for all $x \in \mathbb{R}$.

Let $\mathfrak{h} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. We construct a non-constant continuous periodic function $f : \mathfrak{h} \to \mathbb{C}$ which also satisfies the second property, as follows,

$$f(z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{((8m+5)z + (8n+5))^{2021}}.$$

You may use without proof that this double sum converges absolutely for $z \in \mathfrak{h}$: hence, the value of this double sum does not depend on the order of summation.

c) Show that $f(z) = f(z+8)$ and $f(z) = z^{-2021} f\big(\frac{1}{z}\big)$ for all $z \in \mathfrak{h}$.[1]

Solution.
In this solution, we say that the denominator of a rational number x is the smallest positive integer q for which there exists an integer p such that $x = \frac{p}{q}$. For example, the denominator of $\frac{3}{6}$ is 2.

a) Take for example;

$$f(x) = \begin{cases} q^{2021} & x \in \mathbb{Q} \text{ with denominator } q \\ 0 & \text{else.} \end{cases}$$

b) Write $a = f(0)$. By induction on the size of the denominator q of $x \in \mathbb{Q}$ we prove that $f(x) = aq^{2021}$. Our induction basis holds, as by (i) for integers x we have $f(x) = a = a \cdot 1^{2021}$.

[1]Functions such as in part (c) are called *modular forms*. They play an important role in the proof of *Fermat's last theorem*, as well as in the *sphere packing problem* in dimension 8 and 24.

Now suppose we have proven our claim for all denominators at most equal to q. Let $x \in \mathbb{Q}$ with denominator $q+1$ and write $x = \frac{p}{q+1}$ for some $p \in \mathbb{Z}$. By (i) we can assume that $0 \leq p < q$. Note that $p \neq 0$, as $q > 1$ is the numerator of x and $\frac{0}{q}$ is an integer with denominator equal to 1.

Now, we have

$$f\Big(\frac{p}{q+1}\Big) = \Big(\frac{p}{q+1}\Big)^{2021} f\Big(\frac{q+1}{p}\Big) = \Big(\frac{p}{q+1}\Big)^{-2021} ap^{2021} = a(q+1)^{2021},$$

by which our claim follows.

As for integers n we have that $f\big(\frac{1}{n}\big) = n^{2021}$ and $1/n \to 0$ and $n^{2021} \to \infty$ whenever $n \to \infty$, we conclude that f isn't continuous in 0 unless $a = 0$. In that case $f(x) = 0$ for all $x \in \mathbb{Q}$. The only continuous function $\Gamma \to \Gamma$ with this property is the zero function, but we are looking for a non-constant function.

c) The fact that $f(z) = f(z+8)$ follows by replacing the summation variable n by $n+8m+5$. The transformation for $1/z$ follows by interchanging the summation variables (m, n). As the order of summation doesn't matter, we see that these two transformation are indeed satisfied.

2. Surjective polynomials

dr. R. (Raymond) van Bommel
Massachusetts Institute of Technology

a) Does there exist a $P \in \mathbb{Q}[x]$ such that the function $\mathbb{Q} \to \mathbb{Q}\colon x \mapsto P(x)$ is surjective and $\deg P > 1$?

b) For which prime numbers p does there exist a polynomial $P \in \mathbb{Z}[x]$ such that the function $\mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}\colon x \mapsto P(x) \bmod p$ is surjective and $\deg P > 1$?

c) Does there exist a polynomial $P \in \mathbb{Z}[x]$ with $\deg P > 1$, such that the function $\mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}\colon x \mapsto P(x) \bmod p$ is surjective for infinitely many prime numbers p?

d) Does there exist a polynomial $P \in \mathbb{Z}[x]$ with $\deg P > 1$, such that the function $\mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}\colon x \mapsto P(x) \bmod p$ is surjective for all prime numbers p?

Solution.

a) We will prove that such a polynomial does not exist. Let $P = \sum_{i=0}^{n} c_i x^i \in \mathbb{Q}[x]$ be a polynomial of degree $n \geq 2$. Because P has finitely many coefficients, there is a prime number p such that p does not occur as a prime factor of the numerators and denominators of the coefficients c_i of P. Now we claim that $\frac{1}{p}$ does not lie in the image of P. Suppose that $\frac{a}{b} \in \mathbb{Q}$ is a rational number with $a, b \in \mathbb{Z}$, $\gcd(a,b) = 1$, and $b \neq 0$. Then we have

$$b^n \cdot P\Big(\frac{a}{b}\Big) = \sum_{i=0}^{n} c_i a^i b^{n-i} p^{n-i}.$$

If $p \mid a$ and hence $p \nmid b$, then the right hand side has at least n factors p and $P(\frac{a}{b})$ also needs to have at least n factors p. If $p \nmid a$, then all terms on the right hand side have at least one factor p, except for the term $c_n a^n$ which has 0 factors p. Hence, the left hand side also must have 0 factors p, but this means that the number of factors p in $P(\frac{a}{b})$ has to be a multiple of $n > 1$. In both cases, we conclude that $P(\frac{a}{b}) \neq \frac{1}{p}$, as we wanted to prove.

b) For each prime number p, you can take the polynomial x^p. By Fermat's little theorem, we have $x^p \equiv x \bmod p$ and hence the function is surjective.

c) We will show that the polynomial $P = x^3$ will work. If p is a prime number, then the unit group $(\mathbb{Z}/p\mathbb{Z})^*$ of the field $\mathbb{Z}/p\mathbb{Z}$ is cyclic of order $p-1$. As a function, P maps the element 0 to 0 and the unit group to itself, and this function on the unit group is surjective if and only if $p-1$ is not congruent to 0 mod 3. We will now prove that there are infinitely many prime numbers p such that $p \equiv 2 \bmod 3$.
Suppose there are only finitely many such primes, say $p_1, \ldots, p_\ell$. Then consider the number

$$N = p_1^2 \cdot \ldots \cdot p_\ell^2 + 1 > 1.$$

The number N is congruent to 2 mod 3 and hence the prime factorisation of N cannot only consist of prime numbers which are 0 mod 3 or 1 mod 3, but N is clearly not divisible by by $p_1, \ldots, p_\ell$. This gives a contradiction.

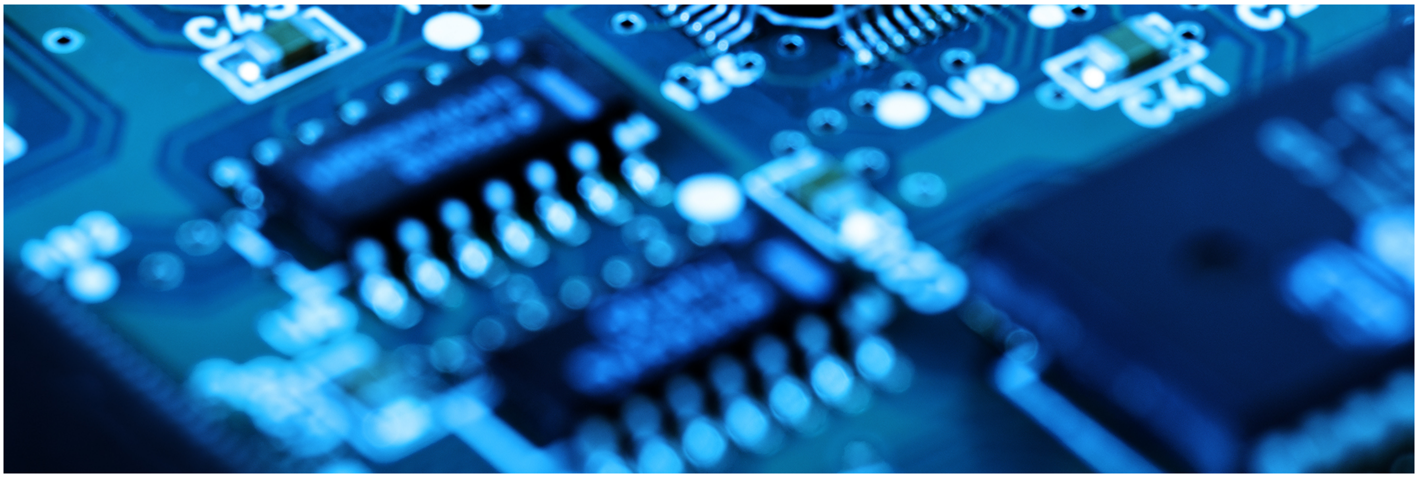
d) We will prove that such a polynomial does not exist. In particular, we will show that there exists an $n \in \mathbb{Z}$ such that $|P(n) - P(n+1)| > 1$. As soon as we proved this, we can take a prime divisor p of $|P(n) - P(n+1)|$ and then it is clear that $P(n) \equiv P(n+1) \bmod p$. In particular, the function $\mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}\colon x \mapsto P(x)$ is not injective, but then it cannot be surjective, as $\mathbb{Z}/p\mathbb{Z}$ is finite.
We prove this by contradiction. Suppose that $|P(n+1) - P(n)| \leq 1$ for all $n \in \mathbb{Z}$. Then we can use the triangle inequality and induction to prove that $|P(n) - P(0)| \leq |n|$ for all $n \in \mathbb{Z}$. If we now write P as $\sum_{i=0}^{d} c_i x^i$ with $d = \deg(P)$ (and hence $c_d \neq 0$), then we see that

$$P(n) - P(0) = \sum_{i=1}^{d} c_i n^i.$$

In particular, we have

$$\lim_{n\to\infty} \frac{P(n) - P(0)}{n} = \lim_{n\to\infty} n^{d-1} \sum_{i=1}^{d} c_i n^{i-d} = \lim_{n\to\infty} n^{d-1} c_d = \begin{cases} \infty & \text{if } c_d > 0, \\ -\infty & \text{if } c_d < 0. \end{cases}$$

This is in contradiction with the inequality $|P(n) - P(0)| \leq |n|$ that we proved.



ASML is a high-tech company, headquartered in the Netherlands. We manufacture the complex lithography machines that chipmakers use to produce integrated circuits, or computer chips. Over 30 years, we have grown from a small startup into a multinational company with over 60 locations in 16 countries and annual net sales of €11.8 billion in 2019.

Behind ASML's innovations are engineers who think ahead. The people who work at our company include some of the most creative minds in physics, electrical engineering, mathematics, chemistry, mechatronics, optics, mechanical engineering, computer science and software engineering.

Because ASML spends more than €2 billion per year on R&D, our teams have the freedom, support and resources to experiment, test and push the boundaries of technology. They work in close-knit, multidisciplinary teams, listening to and learning from each other.

If you are passionate about technology and want to be a part of progress, visit www.asml.com/careers.

3. A quadrant of quadrilaterals

Dr. F. (Fokko) J. van de Bult
Technische Universiteit Delft

Consider a convex quadrilateral $ABCD$ without two parallel sides. Associated to this quadrilateral we define four parallellograms. These are obtained by removing one of the four vertices of the original quadrilateral and creating a parallellogram which contains three of the remaining vertices and which has two sides in common with the original quadrilateral $ABCD$.

For example, with $A = (0, 0)$, $B = (1, 0)$, $C = (2, 1)$, and $D = (3, 5)$, the the parallellogram corresponding to triangle ABC has fourth vertex $D' = (1, 1)$.

Show that exactly one of these four parallellograms is contained completely within the original quadrilateral $ABCD$.

Solution.
Let A' be the point such that $A'BCD$ is a parallellogram. This parallellogram is contained in $ABCD$ when A' is contained in $ABCD$ as $ABCD$ is convex (and the parallellogram consists of all convex linear combinations of A' with points on the sides BC or CD, all of which are then contained in $ABCD$). We will show A' is in $ABCD$ if and only if $\angle B + \angle C > 180°$ en $\angle C + \angle D > 180°$.
First observe that $\angle B + \angle C$ can't be exactly 180° as this would imply AB and CD being parallel. Similarly $\angle C + \angle D \neq 180°$.
Now suppose $\angle B + \angle C < 180°$. Since $A'BCD$ is a parallellogram we have $\angle A'BC + \angle BCD = 180°$. Therefore $\angle A'BC + \angle C = 180°$. Combining gives $\angle B < 180° - \angle C = \angle A'BC$. Thus the line $A'B$ lies inside angle $\langle ABC$, and A' cannot be inside $ABCD$. Similarly from $\angle C + \angle D < 180°$ we obtain that A' is outside the quadrilaterl.
On the other hand, suppose both $\angle B + \angle C > 180°$ and $\angle C + \angle D > 180°$. Then we likewise find that $\langle B$ and $\langle D$ are larger than $\angle A'BC$ and $\angle A'DC$ respectively. Thus the lines $A'B$ and $A'D$ are at least initially in the quadrilateral. Furthermore we observe that $A'B$ and AB intersect in B, $A'B$ and BC also intersect in B and $A'B$ and CD are parallel. Thus $A'B$ can only intersect the quadrilateral in B and on the line AD. So it has to exit the quadrilateral on the line segment AD. Similarly $A'D$ has to go from C to some point on the line segment AB. As $A'B$ is a line that goes from the line segment BC to AD and $A'D$ is a line that goes from the line segment AB to CD. Therefore the intersection point A' of these two lines has to be inside the quadrilateral.
This proves the claim that A' is in $ABCD$ if and only if $\angle B + \angle C > 180°$ en $\angle C + \angle D > 180°$.

In the same way we obtain conditions whether B', C', and D' are in the quadrilateral. These conditions are connected, as the sum of all the angles in a quadrilateral is 360°. For example $\angle B + \angle C > 180°$ is equivalent to $\angle A + \angle D < 180°$. The four conditions for the four points being inside the quadrilateral initially become

$$
\begin{aligned}
A' &: \angle B + \angle C > 180° \wedge \angle C + \angle D > 180° \\
B' &: \angle A + \angle D > 180° \wedge \angle C + \angle D > 180° \\
C' &: \angle A + \angle D > 180° \wedge \angle A + \angle B > 180° \\
D' &: \angle B + \angle C > 180° \wedge \angle A + \angle B > 180°.
\end{aligned}
$$

After rewriting this becomes:

$$\begin{aligned}
A' &: \angle B + \angle C > 180^\circ \wedge \angle C + \angle D > 180^\circ \\
B' &: \angle B + \angle C < 180^\circ \wedge \angle C + \angle D > 180^\circ \\
C' &: \angle B + \angle C < 180^\circ \wedge \angle C + \angle D < 180^\circ \\
D' &: \angle B + \angle C > 180^\circ \wedge \angle C + \angle D < 180^\circ.
\end{aligned}$$

It is now clear that exactly one of these four cases can hold, so always exactly one of the four parallellograms are in the quadrilateral.

Alternative solution: Write $\mathbf{a}$ for the vector from the origin to A, etc. We also assume that the four angles are on the quadrilateral in the order $ABCD$ (counter clockwise). The four parallellograms then have as vertices the endpoints of the vectors

- $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{a}+\mathbf{c}-\mathbf{b}$.
- $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, and $\mathbf{b}+\mathbf{d}-\mathbf{c}$.
- $\mathbf{c}$, $\mathbf{d}$, $\mathbf{a}$ and $\mathbf{c}+\mathbf{a}-\mathbf{d}$.
- $\mathbf{d}$, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{d}+\mathbf{b}-\mathbf{a}$.

For the first new quadrilateral $ABCD'$ the vertex D' is contained in the quadrilateral $ABCD$ when D' is on the same side of the lines CD and AD as B is. We can test this using a determinant, using the well-known lemma below.

Lemma 1. *The points U and V are on the same side of the line XY when*

$$sign(\det(\overrightarrow{XU}\overrightarrow{XY})) = sign(\det(\overrightarrow{XV}\overrightarrow{XY})).$$

(NB This lemma follows directly from the interpretation of the determinant as the area of the parallellogram created by the two vectors in the columns. The sign then shows whether the smallest angle between these two vectors is clockwise or counterclockwise..)
For the quadrilateral $ABCD'$ we thus have the conditions

$$sign(\det(\overrightarrow{CD'}\overrightarrow{CD})) = sign(\det(\overrightarrow{CB}\overrightarrow{CD})), \qquad sign(\det(\overrightarrow{AD'}\overrightarrow{AD})) = sign(\det(\overrightarrow{AB}\overrightarrow{AD})).$$

Due to the orientation of the original quadrilateral we have $\det(\overrightarrow{CB}\overrightarrow{CD}) < 0$ and $\det(\overrightarrow{AB}\overrightarrow{AD}) > 0$. Therefore the conditions become.

$$sign(\det((\mathbf{a}-\mathbf{b})\ (\mathbf{d}-\mathbf{c}))) < 0, \qquad sign(\det((\mathbf{c}-\mathbf{b})\ (\mathbf{d}-\mathbf{a}))) > 0$$

Or more concisely

$$\det((\mathbf{a}-\mathbf{b})\ (\mathbf{d}-\mathbf{c})) < 0 \qquad \wedge \qquad \det((\mathbf{c}-\mathbf{b})\ (\mathbf{d}-\mathbf{a})) > 0$$

Using the other three points we therefore want exactly one of the four following expressions to hold:

$$\begin{aligned}
\det((\mathbf{a}-\mathbf{b})\ (\mathbf{d}-\mathbf{c})) < 0 \qquad &\wedge \qquad \det((\mathbf{c}-\mathbf{b})\ (\mathbf{d}-\mathbf{a})) > 0 \\
\det((\mathbf{b}-\mathbf{c})\ (\mathbf{a}-\mathbf{d})) < 0 \qquad &\wedge \qquad \det((\mathbf{d}-\mathbf{c})\ (\mathbf{a}-\mathbf{b})) > 0 \\
\det((\mathbf{c}-\mathbf{d})\ (\mathbf{b}-\mathbf{a})) < 0 \qquad &\wedge \qquad \det((\mathbf{a}-\mathbf{d})\ (\mathbf{b}-\mathbf{c})) > 0 \\
\det((\mathbf{d}-\mathbf{a})\ (\mathbf{c}-\mathbf{b})) < 0 \qquad &\wedge \qquad \det((\mathbf{b}-\mathbf{a})\ (\mathbf{c}-\mathbf{d})) > 0
\end{aligned}$$

Observe that all the time we consider the same two determinants; only the signs are flipped due to the interchange of some vectors, or multiplying one or both of the vectors by -1. If we set

$$\delta_1 = \det((\mathbf{a} - \mathbf{b})\ (\mathbf{c} - \mathbf{d})), \qquad \delta_2 = \det((\mathbf{a} - \mathbf{d})\ (\mathbf{b} - \mathbf{c}))$$

than the conditions become

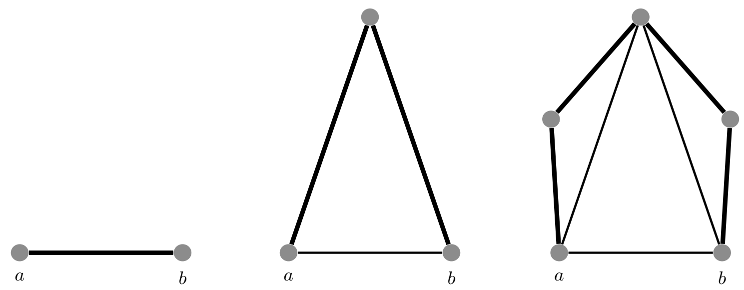
$$\begin{array}{ccc} \delta_1 > 0 & \wedge & \delta_2 < 0 \\ \delta_2 > 0 & \wedge & \delta_1 > 0 \\ \delta_1 < 0 & \wedge & \delta_2 > 0 \\ \delta_2 < 0 & \wedge & \delta_1 < 0 \end{array}$$

Since all four the combinations of signs occur precisely once, it is clear that exactly on of these four options hold. Note that the determinants cannot be 0, as the sides are not allowed to be parallel.

Remark: Consider a general third degree polynomial in two variables $p(x, y) = ax^3 + bx^2y + \cdots + j$. It has 10 degrees of freedom for the variables a through j. The condition that a point (x, y) is a critical point ($\nabla p = \mathbf{0}$), gives two linear equations in these 10 variabeles. Of course, j never occurs, as it will be differentiated away. You can also multiply a polynomial by any non-zero constant without changing the critical points. Therefore, if we specify four points as being critical we obtain 8 equations in the 9 variables a through i. Generically we have a 1-dimensional solution set, corresponding to a unique polynomial with these critical points up to scaling.
If the four critical points form a convex quadrilateral, it turns out two of the critical points will be saddle points, and the other two are the location of a (local) maximum and minimum. Using the geometry of this exercise you can determine which are the saddle points: If D' is contained in the quadrilateral $ABCD$, then A and C are saddle points, and B and D are the locations of the maximum and minimum. And if C' is contained in $ABCD$, then B and D are the saddle points, etc.

4. Fringed tulip

dr. ir. Rik Versendaal en dr. Ivan Kryven,
Universiteit Utrecht



0-tulip 1-tulip 2-tulip

An n-tulip is the following iterative construction: A 0-tulip consist of two initial vertices a and b connected with a link. Iteratively, an $(n+1)$-tulip is obtained from an n-tulip by glueing a triangle to each newly added link at the previous iteration.

A fringed n-tulip is an n-tulip in which each link is removed with probability $1-p$. We say that a and b are connected with a path, if there is at least one way to travel from a to b by following the links. We are interested in $f_n(p)$, the probability that there is a path from a to b in a fringed n-tulip. Note that by definition, $f_0(p) = p$, because the only possible path is the link (a, b) itself.

Show that:

a) $f_n \in C^\infty(0, 1)$ for $n \in \mathbb{N}_0$.

b) $f_n(p)$ converges for all $p \in (0, 1)$.

Let $F : (0, 1) \to [0, 1]$ be defined by $F(p) := \lim_{n\to\infty} f_n(p)$. Show that:

c) $F \in C^0(0, 1)$ and $F \notin C^1(0, 1)$.

Solution.
By definition, we have

$$f_0(p) = p.$$

In 1-Tulip, there are two paths: one may take the initial link, which is present with probability $f_0(p) = p$, or if it is not present then there is still a detour of two links, which are simultaneously present with probability p^2. Hence,

$$f_1(p) = p + (1-p)p^2,$$

and, in general, we have a recursion:

$$f_{n+1}(p) = A_p(f_n(p)) := p + (1-p)f_n(p)^2.$$

One can see that $f_n(p)$ is a polynomial of order $2n-1$, and therefore, it is infinitely differentiable when n is finite, **which answers a)**.

To prove that $f_n(p)$ converges, first note that $f_1(p) > f_0(p)$. Assuming that $f_n(p) > f_{n-1}(p)$ we have

$$f_{n+1}(p) = p + (1-p)f_n(p)^2 > p + (1-p)f_{n-1}(p)^2 = f_n(p).$$

By induction, it follows that $f_n(p)$ is monotonically increasing sequence. Since also $f_n(p) \leq 1$ for all n, it follows that $f_n(p)$ converges. **This answers b)**.

We compute $F(p)$. Since $A_p(x)$ is continuous, we know that $F(p)$ is a fixed point of A_p. Note that the equation

$$A_p(x) = x$$

has two roots, namely $x_1^* = 1$ and $x_2^* = \frac{p}{1-p}$.
To determine $F(p)$ we study the cases of $p > \frac{1}{2}$ and $p \leq \frac{1}{2}$ separately.

- Let $p > \frac{1}{2}$. Since $f_n(p) \leq 1$ we must have that $F(p) \leq 1$. Because, $p > \frac{1}{2}$, we have $x_2^* = \frac{p}{1-p} > 1$. Therefore, we find that $F(p) = x_1^* = 1$.

- Let $p \leq \frac{1}{2}$. In that case, we have $x_1^* \geq x_2^*$. Since $p < \frac{1}{2}$, we have $f_0(p) = p \leq \frac{p}{1-p}$. Assuming that $f_n(p) \leq \frac{p}{1-p}$, we find that

  $$f_{n+1}(p) = p + (1-p)f_n(p)^2 \leq p + (1-p)\frac{p^2}{(1-p)^2} = \frac{p}{1-p}.$$

  Using indunction, we conclude that $f_n(p) \leq \frac{p}{1-p} = x_2^*$ for all n. This implies that $F(p) \leq x_2^*$. Since $x_2^* \leq x_1^*$, it follows that $F(p) = x_2^* = \frac{p}{1-p}$.

Bringing these two cases together, we have

$$F(p) = \begin{cases} \frac{p}{1-p}, & p < \frac{1}{2}, \\ 1, & p \geq \frac{1}{2}, \end{cases}$$

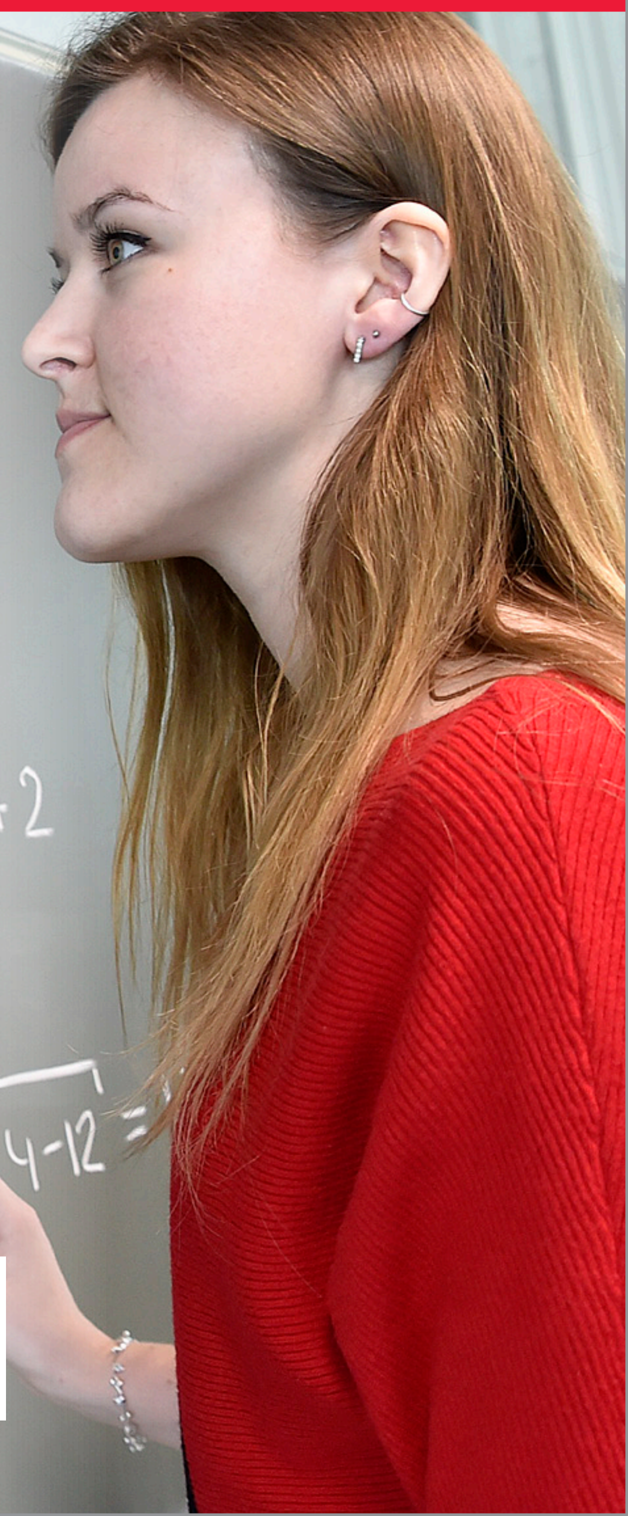
which is everywhere continuous, but not differentiable at $p = \frac{1}{2}$, **hence showing c)**.

university of groningen

faculty of science and engineering

International Master's degree programme in

• Mathematics
• Applied Mathematics



- Abstract thinking
- Logical reasoning and mathematical proof
- Explaining the world around you using mathematical models
- Design and simulation
- Problem-solving & analysis

TRACKS

Mathematics

- Statistics and Big Data
- Mathematics and Complex Dynamical Systems

Applied Mathematics

- Systems and Control
- Computational Mathematics

www.rug.nl/masters/mathematics

www.rug.nl/masters/applied-mathematics

5. From sumrise to sumset

dr. H. J. (Harry) Smit en M. (Merlijn) Staps, MSc.
Max Planck Institute Bonn & Princeton University

Although the LIMO's problems (*sommen*) are different each year, in this problem we will consider sets whose sums (*sommen*) are all equal.
Suppose we have some red and blue cards with an integer number on each card, such that for every integer k the number of ways to select a number of red cards with sum k equals the number of ways to select a number of blue cards with sum k.

a) Prove that if the cards contain only positive numbers, then for every positive integer ℓ it holds that the number of red cards containing ℓ equals the number of blue cards containing ℓ.

b) Prove that if the cards may also contain negative integers, then for every positive integer ℓ it holds that the number of red cards containing ℓ or $-\ell$ equals the number of blue cards containing ℓ or $-\ell$.

(c) Prove that if the red and blue cards do not contain exactly the same numbers (i.e., the number of red cards containing ℓ differs from the number of blue cards containing ℓ for at least one integer ℓ), then we can select a positive number of red cards such that the sum of the numbers on these cards equals 0.

In this problem you are allowed to use the results of earlier parts, even when you have not yet solved them.

Solution.

a) Notice that there are equal numbers of red and blue cards, because the total number of ways to select a number of red cards equals $2^{|A|}$ (where A denotes the set of red cards) and the number of ways to select a number of blue cards equals $2^{|B|}$ (where B denotes the set of blue cards). Order both the red and blue cards by their numbers, from small to large, and suppose that the m-th red card is the first one that differs from the corresponding m-th blue card (if such an m does not exist, we are immediately done). Without loss of generality, we assume that the number x on the m-th red card is smaller than the number on the corresponding m-th blue card. When we choose a number of blue cards with sum x, we can only use the first $m-1$ cards, because the other blue cards contain numbers larger than x. For each of these choices, there is a corresponding choice of red cards that also has sum x, because the numbers on the first $m-1$ red cards are equal to the numbers on the first $m-1$ blue cards. However, there is at least one additional way to choose some red cards with sum x: by choosing only the m-th card. We conclude that the condition in the problem is not satisfied for $k = x$. This contradiction completes the proof.

b) Let n_A be the sum of the numbers on the red cards that contain a negative number, and define n_B analogously for the blue cards. Because n_A is the smallest integer that can be written as the sum of numbers on red cards, and similarly n_B is the smallest integer that can be written as the sum of numbers on blue cards, we must have $n_A = n_B$.

For every red card containing a number x, we create a corresponding orange card containing $|x|$. Similarly, for every blue card containing a number y, we create a corresponding purple card containing $|y|$. It suffices to show that for every positive integer ℓ

the number of orange cards containing ℓ equals the number of purple cards containing ℓ. Because the orange and purple cards contain only positive integers, we can use part a) for this. We have to prove that the orange and purple cards have the property that for every integer k the number of ways to select orange cards with sum k equals the number of ways to select purple cards with sum k. Suppose that we have chosen some orange cards with sum k. Every orange card corresponds to a red card. We now consider those red cards for which

- the card contains a positive integer, and the corresponding orange card is chosen; or,
- the card contains a negative integer, and the corresponding orange card is not chosen.

Among the red cards that contain a positive integer we have now chosen those red cards for which the corresponding orange card has been chosen. For each red card containing a negative number, we now either have chosen the red card but not the corresponding orange card, or we have chosen the corresponding orange card but not the red card. It follows that the sum of the chosen red cards equals k plus the sum of all the negative numbers that occur on the red cards; hence, the sum of the chosen red cards is $k + n_A$. Moreover, our procedure by which we link a choice of orange cards to a choice of red cards is bijective. It follows that the number of ways to select orange cards with sum k equals the number of ways to select red cards with sum $k + n_A$. Analogously, the number of ways to select purple cards with sum k equals the number of ways to select blue cards with sum $k+n_B$. From the fact that $n_A = n_B$ together with the fact that the blue and red cards satisfy the condition in the problem, it now follows that the number of ways to select orange cards with sum k equals the number of ways to select purple cards with sum k, as desired.

c) We begin by throwing away red and blue cards containing the same number, until the numbers on the red cards are different from the numbers on the blue cards. Then the problem condition is still satisfied, and at least one card is left. Moreover, from part b) it follows that the numbers on the blue cards are the (additive) inverses of the numbers on the red cards. Let s the sum of the red cards containing positive numbers. Then the sum of the blue cards containing negative numbers is $-s$. Because $-s$ is the smallest number that can be written as a sum of blue cards, $-s$ must also be the smallest number that can be written as a sum of red cards. It follows that the sum of the red cards containing negative numbers equals $-s$. Therefore, the sum of all remaining red cards (i.e., the red cards that we did not throw away) equals 0.

6. A pretty problem on positive polynomials

M. (Mike) Daas, MSc.
Universiteit Leiden

Let P be a polynomial with positive coefficients. Determine the maximum of $P(x)^2/P(x^2)$ over all $x \in \mathbb{R}$. For which x is this maximum attained?

Solution.
Write $P(x) = \sum_{i=0}^{n} a_i x^i$ and consider the vectors

$$(\sqrt{a_0}, \sqrt{a_1}, \ldots, \sqrt{a_n}) \quad \text{en} \quad (\sqrt{a_0}, \sqrt{a_1}x, \ldots, \sqrt{a_n}x^n).$$

Applying the Cauchy-Schwarz inequality, we find that

$$(a_0 + a_1 + \ldots + a_n)(a_0 + a_1 x^2 + \ldots a_n x^{2n}) \geq (a_0 + a_1 x + \ldots + a_n x^n)^2.$$

In other words, we find that $P(1)P(x^2) \geq P(x)^2$ and so $P(x)^2/P(x)^2 \leq P(1)$. Equality can only occur when the two considered vectors are scalar mutliples of each other. Their first components agree and are non-zero, so the vectors must be equal. This means that only in $x = 1$ the maximum can be attained.

Over het Koninklijk Wiskundig Genootschap (KWG)

Wat is het KWG? In 1778 opgericht, beoogt het KWG een verbindend orgaan te zijn voor de wiskundige beroepsgroep en een stimulans te bieden voor wiskundige activiteiten. Daarnaast vormt het KWG samen met de Nederlandse Vereniging voor Wiskundeleraren de twee pijlers van het Platform Wiskunde Nederland (PWN) dat politieke belangen van wiskundig Nederland behartigt.

Activiteiten van het KWG zijn o.a.:

- Uitbrengen van Pythagoras, een wiskundetijdschrift voor scholieren.
- Ondersteunen van de Kaleidoscoopdagen (die georganiseerd worden door de studieverenigingen).
- Organisatie van het jaarlijkse Nederlands Mathematisch Congres (voor alle wiskundigen in Nederland, i.h.b. voor wiskundigen werkzaam aan de universiteiten).
- Organisatie van het Wintersymposium (voor wiskundeleraren).
- Uitbrengen van Nieuw Archief voor de Wiskunde (4x per jaar voor alle leden; met informatie en artikelen over wiskunde voor algemeen wiskundig publiek).
- Uitbrengen van Indagationes Mathematicae (een internationaal wetenschappelijk tijdschrift).
- Beheren van Nederlandse wiskundige nalatenschap, bijv. het archief van Brouwer.

Wat kan het KWG betekenen voor wiskundestudenten?

- Één jaar gratis lidmaatschap (m.a.w.: 4x gratis het Nieuw Archief voor de Wiskunde.)
- Korting op het lidmaatschap zo lang je studeert.
- Goedkoop bijwonen Nederlands Mathematisch Congres.

Wie zit in het bestuur van het KWG? (anno najaar 2019) Danny Beckers (VU), Theo van den Bogaart (HU), Sonja Cox (UvA), Marije Elkenbracht (ABN AMRO), Barry Koren (TUE), Marie-Colette van Lieshout (CWI/UT), Michael Műger (RU), Wioletta Ruszel (UU) en Jan Wiegerinck (UvA).

De bestuursleden zijn tevens aanspreekpartners op de verschillende universiteiten.

Vragen? Kijk op de website: wiskgenoot.nl, of neem contact op met de secretaris: secretaris@wiskgenoot.nl.

7. Mirror reflex triangles

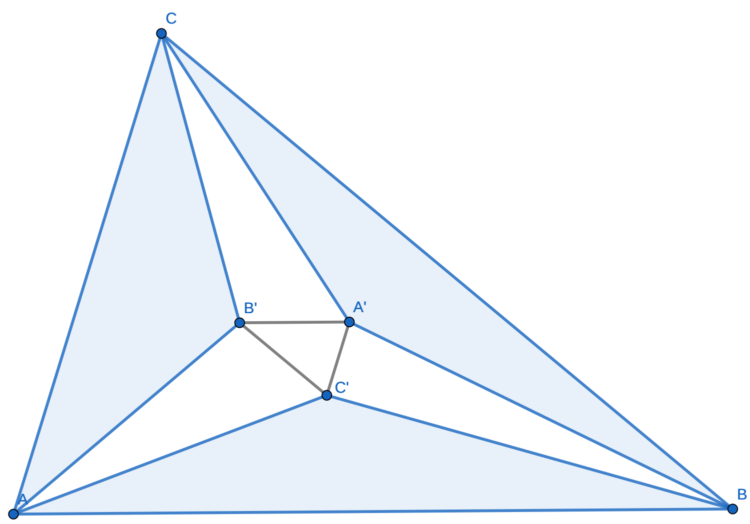
Ir. H. (Harold) de Boer
Transtrend BV

Let ABC be a triangle. Somewhere within triangle ABC there is a smaller triangle $A'B'C'$ such that:

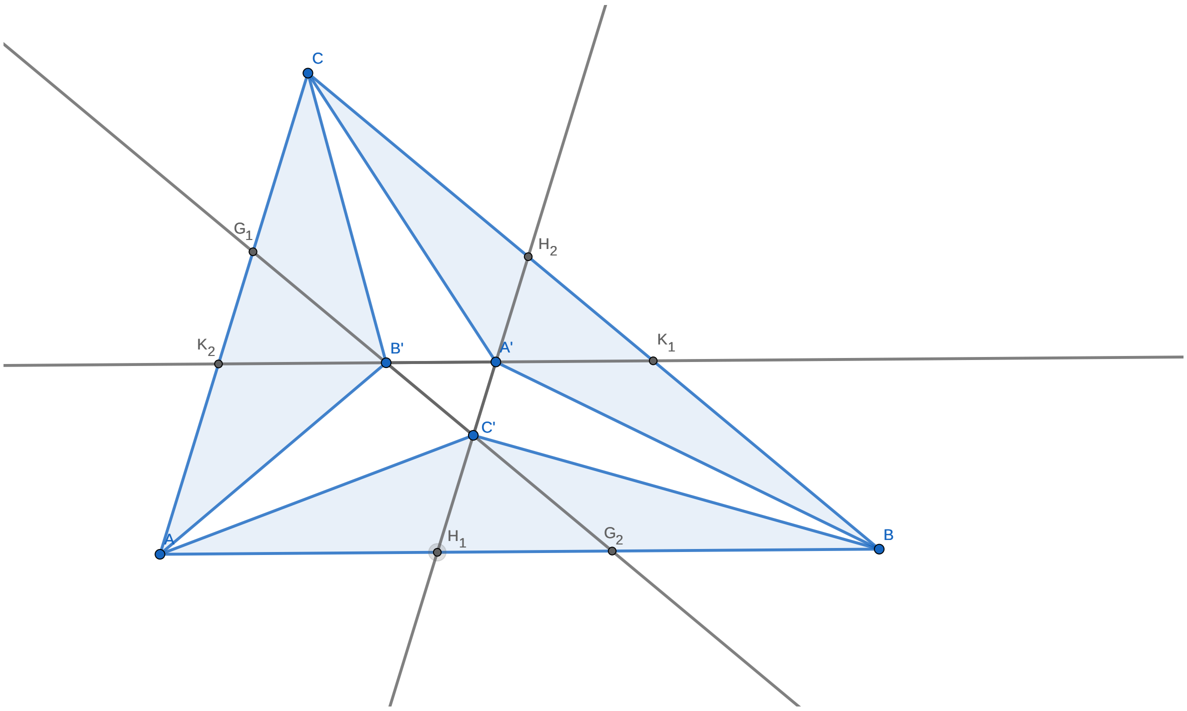
- $A'B'$ is parallel to AB,
- $B'C'$ is parallel to BC,
- $C'A'$ is parallel to CA,
- The surface area of $A'B'C'$ is f^2-times that of ABC.

Determine the cumulative surface area of the triangles BCA', CAB' and ABC' as a fraction of the surface area of ABC, expressed in terms of f.

Note: Your solution should hold for any triangle ABC, irrespective of the exact location of $A'B'C'$ within ABC.



Solution.
From the properties above, it follows immediately that $A'B'C'$ and ABC are similar triangles. The edges of $A'B'C'$ are f-times those of ABC. Extend $B'C'$ such that it intersects with AC and AB. Let G_1 be the intersection of the extension of $B'C'$ with AC and let G_2 be the intersection of the extension of $B'C'$ with AB. Similarly, we can define H_1 and H_2 respectively as the intersections given by extending $C'A'$ through AB and BC. Extending $A'B'$ through BC and AC yields K_1 and K_2, respectively.



All triangles AG_2G_1, H_1BH_2 and K_2K_1C are similar to ABC, with their edges smaller by some factors g, h, and k, respectively, than those of ABC. Any triangle $A'B'C'$ within ABC with the properties as listed above is uniquely determined by the factors g, h and k.

Furthermore, all triangles H_1G_2C', $A'K_1H_2$ and $K_2B'G_1$, are similar to ABC, with their edges smaller by a factor $(g + h - 1)$, $(h + k - 1)$ and $(k + g - 1)$, respectively, than those of ABC. From the latter it follows that ABC', BCA' and CAB' have surface areas equal to a fraction $(g + h - 1)$, $(h + k - 1)$ and $(k + g - 1)$, respectively, of the total surface area of ABC. Therefore, the total surface area of these triangles H_1G_2C', $A'K_1H_2$ and $K_2B'G_1$ as a fraction of that of ABC equals $2(g + h + k) - 3$.

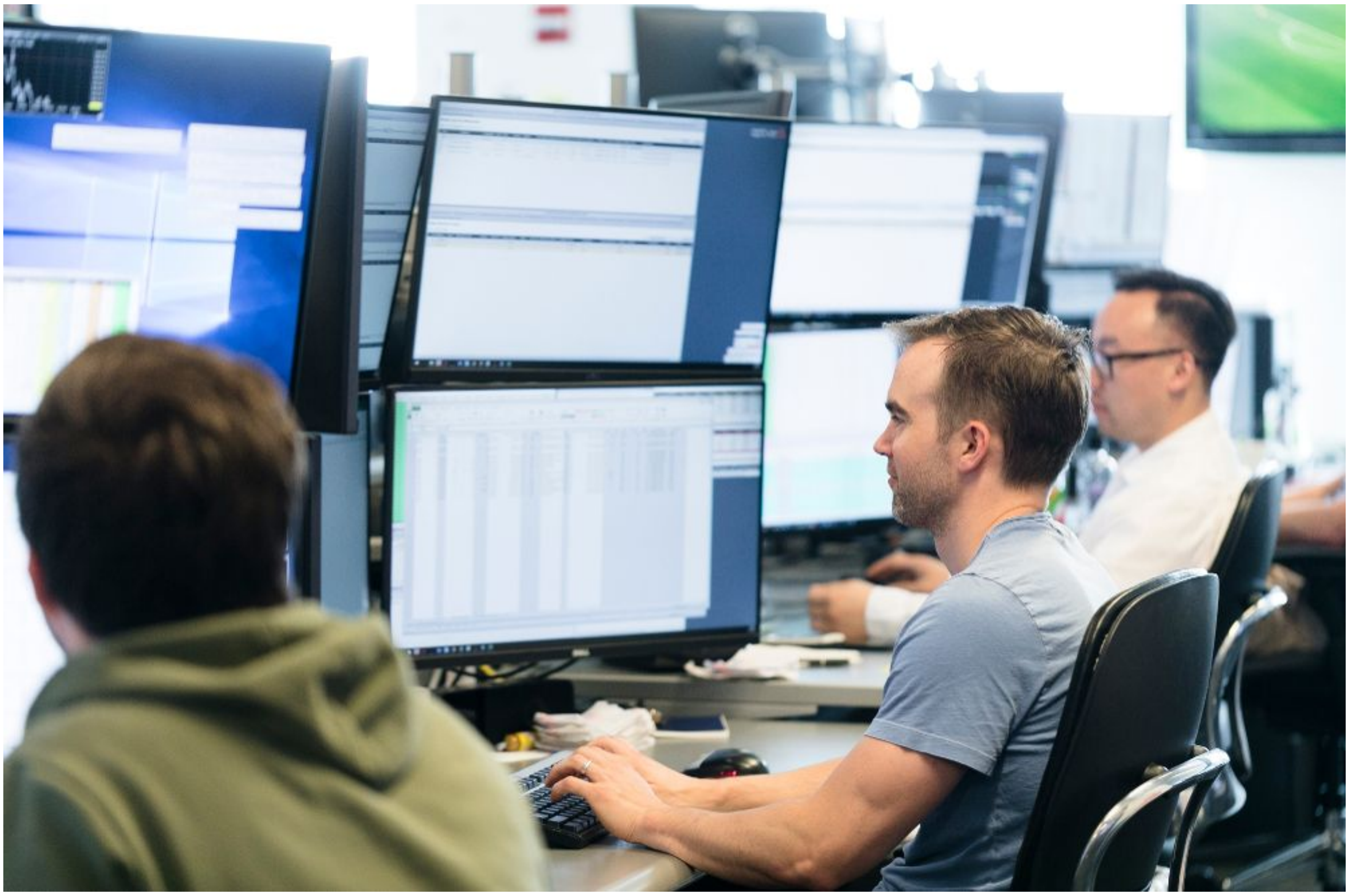
The missing link is a relation between f and g, h and k. Note that: $|B'C'| = |G_1G_2| - |G_1B'| - |C'G_2|$. This yields $f|BC| = g|BC| - (g+k-1)|BC| - (k+h-1)|BC|$ which implies that $f = 2 - (g + h + k)$ and thus $g + h + k = 2 - f$. Substituting the total surface area above into this equation yields $2(g + h + k) - 3 = 2(2 - f) - 3 = 1 - 2f$.

8. Sums of invertable matrices

Prof. dr. H. W. (Hendrik) Lenstra
Universiteit Leiden

Let n be a positive integer, K be a field, and A be an $n \times n$-matrix over K that is not the sum of two invertable $n \times n$-matrices over K. Prove that $n = 1, \#K = 2, A = (1)$.

Solution.
For $n = 1$ this is easy. Therefore assume $n > 1$. In this case it suffices to show that if V and W are two K-vector spaces of dimension n, then every K-linear map $f : V \to W$ is the sum of two isomorfisms. To do this, we choose a basis $f(v_1), \ldots, f(v_k)$ of $f(V)$, with v_i in V. Then we can fill $v_1, \ldots, v_k$ with a basis $v_{k+1}, ..., v_n$ of the kernel of f to a basis $v_1, ..., v_n$ of V. We can also fill $w_n = f(v_1), w_1 = f(v_2), ..., w_{k-1} = f(v_k)$ to a basis $w_1, ..., w_n$ of W. Now let F be the $n \times n$-matrix that describes f on both these bases; then F is a matrix with k ones outside of the diagonal and for the rest zeros. In particular, F is a matrix that has only zeros on the diagonal. Let B now be the matrix that has ones everywhere on the diagonal, above the diagonal is the same as F and below the diagonal has only zeros. Then $C = F - B$ has everywhere on de diagonal -1, is the same as F below the diagonal and is zero above the diagonal. So we have $F = B + C$ with B and C both invertable, and $f = b + c$ with b and c isomorfisms from V to W.



What does trading mean? How could you use your mathematical and technical skills in the financial markets?

Join our lunch lecture with Prasad Chebolou, one of our Quantitative Researchers, who will join you tell you about trading and how you could use your mathematical skills at this industry.

We are market makers. In simple terms, we provide buy and sell prices for the financial products in exchanges all over the world. We help keep the markets viable by creating the liquidity needed to allow everyone to trade at will.

Interested to learn more about how you can use your skills in our company? Join us at 12pm to get the insights!

optiver

9. Superpositions of partitions

dr. M. (Martijn) Kool
Universiteit Utrecht

A *partition* is a sequence of non-negative integers $\lambda = \{\lambda_i\}_{i>0}$, such that $\lambda_i > 0$ for finitely many i and $\lambda_i \geq \lambda_{i+1}$ for all $i > 0$. We call $|\lambda| := \sum_i \lambda_i$ the *size* of λ. Define Λ to be the collection of partitions with positive size

a) Prove that

$$1 + \sum_{\lambda \in \Lambda} q^{|\lambda|} = \prod_{n=1}^{\infty} \frac{1}{1-q^n}.$$

The *support* of a partition λ is the function $f_\lambda : \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \to \{0,1\}$ with

$$f_\lambda(i,j) := \begin{cases} 1 & \text{if } j \leq \lambda_i \\ 0 & \text{else.} \end{cases}$$

A *flat partition* is a sequence of non-negative integers $\pi = \{\pi_{ij}\}_{i,j>0}$ such that $\pi_{ij} > 0$ for finitely many i,j and $\pi_{ij} \geq \pi_{i+1,j}$,$\pi_{ij} \geq \pi_{i,j+1}$ for all $i,j \geq 1$. We call $|\pi| := \sum_{i,j} \pi_{ij}$ the *size* of π. Define Π to be the collection of all flat partitions with positive size. For a flat partition $\pi \in \Pi$ we assign a weight w_π as follows. Set

$$W_\pi := \Big\{ \{n_\lambda\}_{\lambda \in \Lambda} : n_\lambda \in \mathbb{Z}_{\geq 0} \ \forall \lambda \in \Lambda \text{ en } \pi_{ij} = \sum_{\lambda \in \Lambda} n_\lambda \cdot f_\lambda(i,j) \ \forall i,j \geq 1 \Big\},$$

and define the weight of π as

$$w_\pi := \prod_{\{n_\lambda\}_{\lambda \in \Lambda} \in W_\pi} \prod_{\lambda \in \Lambda} \frac{1}{n_\lambda !}.$$

b) Prove

$$1 + \sum_{\pi \in \Pi} w_\pi p^{\pi_{11}} q^{|\pi|} = \exp\Big(p \sum_{\lambda \in \Lambda} q^{|\lambda|} \Big).$$

c) Prove

$$1 + \sum_{\pi \in \Pi} w_\pi \prod_{n=1}^{\pi_{11}} (N - (n-1)) q^{|\pi|} = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^N},$$

for all $N \in \mathbb{Z}_{>0}$.

Solution.

a) We recognize the expression of the geometrical series

$$\frac{1}{1-q^n} = 1 + q^n + q^{2n} + q^{3n} + \dots.$$

So for the product we find

$$\prod_{n=1}^{\infty} \frac{1}{1-q^n} = (1 + q + q^2 + \dots)(1 + q^2 + q^4 + \dots)(1 + q^3 + q^6 + \dots) \cdots.$$

Now we will make a bijection between Λ and the non-1-terms of this expression. With a term we mean something of the form $q^{m_1+2m_2+3m_3+\dots}$ where $m_i \in \mathbb{Z}_{\geq 0}$ and the term is thus obtained by multiplying from $(1+q+q^2+\dots)$ the m_1st term (q^{m_1}) with from $(1+q^2+q^4+\dots)$ the m_2st term (q^{2m_2}) et cetera.
So, assume that $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ and define r_m as the number of λ_i equal to m. Then the bijection is given by

$$\lambda \mapsto q^{r_1} q^{2r_2} q^{3r_3} \cdots = q^{r_1+2r_2+3r_3+\dots} = q^{|\lambda|}.$$

This indeed gives a bijection from which we directly see

$$\sum_{\lambda\in\Lambda} q^{|\lambda|} = \left(\prod_{n=1}^{\infty} \frac{1}{1-q^n}\right) - 1.$$

Taking the 1 to the other side proves the exercise.

b) We start by rewriting the rightside. For this we use first that $\exp(x+y) = \exp(x)\exp(y)$ and then that the Taylor series of $\exp(x)$ equals $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$.

$$\exp\Big(p\sum_{\lambda\in\Lambda} q^{|\lambda|}\Big) = \prod_{\lambda\in\Lambda} \exp(pq^{|\lambda|}) = \prod_{\lambda\in\Lambda} \big(1 + pq^{|\lambda|} + \frac{p^2q^{2|\lambda|}}{2!} + \dots\big).$$

Secondly we note that $\pi_{11} = \sum_{\lambda\in\Lambda} n_\lambda$ and $|\pi| = \sum_{\lambda\in\Lambda} n_\lambda|\lambda|$. With this the left side equals

$$1 + \sum_{\pi\in\Pi}\Big(\prod_{\{n_\lambda\}_{\lambda\in\Lambda}\in W_\pi}\prod_{\lambda\in\Lambda}\frac{1}{n_\lambda!}p^{n_\lambda}q^{n_\lambda|\lambda|}\Big).$$

Now we will explain why these two expressions are the same.
Consider a term of the first expression that is the product of $\frac{p^{m_\lambda}q^{m_\lambda}}{m_\lambda!}$ over alle λ. By taking $n_\lambda = m_\lambda$ we then also get a unique term of the second expression. Note for this that each $\{n_\lambda\}_{\lambda\in\Lambda}$ give a unique π. Here we only miss the sequence with $m_\lambda = 0$ for all λ but that one gets compensated by the 1 at the beginning.

c) Note that $w_\pi \prod_{n=1}^{\pi_{11}}(N-(n-1))q^{|\pi|}$ equals 0 for π with $\pi_{11} \geq N+1$. So the sum is taken over the π for which we have $\pi_{11} \leq N$. The rightside can be rewritten with a) to become:

$$\prod_{n=1}^{\infty}\frac{1}{(1-q^n)^N} = \big(1 + \sum_{\lambda\in\Lambda} q^{|\lambda|}\big)^N.$$

We consider a term of this expression to be something of the form

$$q^{|\lambda_1|}q^{|\lambda_2|}\cdots q^{|\lambda_N|}.$$

Where we also take the sequece $(0,0,\dots,0)$ such that $q^{|\lambda|} = 1$. Now let

$$n_\lambda = \#\{i \leq N \text{ zodat } \lambda = \lambda_i\}.$$

Assume that this gives us the set $\{n_{\lambda'_1}, \dots, n_{\lambda'_m}\}$ of $n_\lambda \neq 0$. Then we can get this combination $\{\lambda_1, \dots, \lambda_N\}$ in

$$\frac{N!}{(n_{\lambda'_1})!\cdots(n_{\lambda'_m})!}$$

ways.
Now we can rewrite the left side of this expression. For this we again use that $\pi_{11} = \sum_{\lambda\in\Lambda} n_\lambda$ and $|\pi| = \sum_{\lambda\in\Lambda} n_\lambda|\lambda|$. We get

$$1 + \sum_{\pi\in\Pi} \prod_{\{n_\lambda\}_{\lambda\in\Lambda}\in W_\pi} \prod_{\lambda\in\Lambda} \frac{1}{n_\lambda!} \prod_{n=1}^{\sum n_\lambda} (N - (n-1)) q^{\sum n_\lambda|\lambda|}.$$

This can be rewritten to

$$1 + \sum_{\pi\in\Pi} \prod_{\{n_\lambda\}_{\lambda\in\Lambda}\in W_\pi} \frac{N!}{(\prod_{\lambda\in\Lambda} n_\lambda!)(N - \sum n_\lambda)!} q^{\sum n_\lambda|\lambda|}.$$

In a same manner as at b) we can prove that these expressions are equal.

10. Relative prime count is not relative prime

drs. S. (Stijn) Cambie
Radboud Universiteit Nijmegen

The Euler totient function ϕ is defined by mapping a positive integer n with prime factorization $\prod_{i=1}^{j} p_i^{e_i}$ to

$$\prod_{i=1}^{j} (p_i - 1)p_i^{e_i - 1}.^2$$

What is the smallest ratio m/n such that for all positive integers k, we have $\phi(k!)^n \mid (k!)^m$?

Solution.
We will prove that this ratio is $\frac{7}{4}$ and this is the smallest possible due to the example $k = 7$. Let $v_p(n)$ denote the exponent of p in the prime factorization of n. It is enough to prove that for every prime p, we have $4v_p(\phi(k!)) \leq 7v_p(k!)$.
For $k < p$ it is trivial and for $k \geq p$, we have

$$v_p(\phi(k!)) = v_p(k!) - 1 + v_p\left(\prod_{q \leq k \ prime, q \equiv 1 \pmod p} (q-1)\right).$$

Now we first prove that the key is to look to $p = 2$.

Lemma 2. *For p odd, we have $\frac{3}{2}v_p(k!) \geq v_p(\phi(k!))$ for every k.*

Proof. Note that it is enough to prove that

$$v_p\left(\prod_{q \leq k \ prime, q \equiv 1 \pmod p} (q-1)\right) \leq \frac{1}{2}v_p(k!).$$

But since p is odd, we need $q \equiv 1 \pmod{2p}$. This implies that

$$\begin{aligned} v_p\left(\prod_{q \leq k \ prime, q \equiv 1 \pmod p} (q-1)\right) &\leq v_p\left(2p \cdot 4p \cdot \ldots \cdot 2p\left\lfloor \frac{k}{2p} \right\rfloor\right) \\ &= v_p\left(p \cdot 2p \cdot \ldots \cdot p\left\lfloor \frac{k}{2p} \right\rfloor\right) \\ &= v_p\left(\left\lfloor \frac{k}{2} \right\rfloor !\right) \\ &\leq \frac{1}{2}v_p(k!). \end{aligned}$$

The last inequality is a consequence of $\binom{k}{\lfloor \frac{k}{2} \rfloor}$ being an integer. □

Now we will deal with $p = 2$.

Lemma 3. *For every k, we have $v_2(k!) \leq k - 1$.*

Proof. Note that

$$v_2(k!) = \sum_{i \geq 1} \left\lfloor \frac{k}{2^i} \right\rfloor < \sum_{i \geq 1} \frac{k}{2^i} = k$$

from which the lemma follows. □

[2]This is equal to the number of integers between 0 and n realtively prime to n.

Note that it is enough to prove that for odd k, we have

$$v_2\left(\prod_{q\le k\ prime, q\equiv 1 \pmod 2}(q-1)\right)\le \frac{3}{4}v_2(k!)+1.$$

This is equivalent with proving that for odd k, we have

$$v_2\left(\prod_{q\le k\ composite}(q-1)\right)\ge \frac{1}{4}v_2(k!)-1.$$

This is easily seen to be true for $k\le 7$, with equality if $k=7$.
For odd $k\ge 9$, we can even prove that

Lemma 4. *For odd $k\ge 9$, we have*

$$v_2\left(\prod_{q\le k\ composite}(q-1)\right)\ge \frac{1}{4}k-1.$$

Proof. Induction basis: This is easily checked for $9\le k\le 21$ since $9,15$ and 21 are composite.
Induction hyposthesis: Assume it is proven up to $9\le k\le K$ where $K\ge 21$.
Induction step: We prove it for $K+2$.

$$\begin{aligned}v_2\left(\prod_{q\le K+2\ composite}(q-1)\right)&\ge v_2\left(\prod_{q\le K-10\ composite}(q-1)\right)+3\\ &\ge \frac{K-10}{4}-1+3=\frac{K+2}{4}-1.\end{aligned}$$

since there are at least two composite numbers $K-10<q\le K+2$, one being $\equiv 3 \pmod{12}$ and an other one being $\equiv 9 \pmod{12}$. So by complete induction, the statement is true for all odd $k\ge 9$. □

So by taking into account Lemma 3 we conclude.

transtrend

Probleem:

Een getal is een tweemacht wanneer het kan worden geschreven als 2^k met k een geheel getal ≥ 0.

Een getal is een reekssom als het de som is van een reeks van minimaal 2 opeenvolgende positieve gehele getallen, bijvoorbeeld $15 = 4 + 5 + 6$.

Bewijs dat alle positieve gehele getallen of een tweemacht zijn, of een reekssom, maar nooit beide.

Stuur je bewijs naar wiskunde@transtrend.com

"Bij Transtrend ontwikkelen vindingrijke bèta's systematische handelsstrategieën waarmee het vermogen van professionele beleggers wordt beheerd."

11. Real periodic orbits

Prof. dr. J. (Jaap) Top
Rijksuniversiteit Groningen

Starting from a polynomial $p(x)$ in one variable x with $p(x)$ having real coefficients, and a complex number a_1, one defines a sequence $(a_n)_{n\geq 1}$ by iterating: so $a_2 = p(a_1)$, $a_3 = p(a_2)$ et cetera: $a_{m+1} = p(a_m)$ for every integer $m \geq 1$.

The sequence $(a_n)_{n\geq 1}$ is called a periodic orbit of the iteration, if $a_m = a_1$ for some $m > 1$. The periodic orbit is called real, if moreover $a_n \in \mathbb{R}$ for every n. As a simple example, take $p(x) = x^2$. Then every sequence starting with $a_0 = e^{2\pi i r}$ for some rational number r with an odd denominator is a periodic orbit. Also the sequence starting with $a_0 = 0$ is periodic. It turns out that these are the only periodic orbits for this polynomial, so in particular the only real periodic orbits in this case are the ones starting with $a_1 \in \{0, 1\}$. Many of periodic orbits for the given polynomial are not real.

The situation is very different for the polynomial $q(x) = 2x^2 - 1$: Show that for $q(x)$ all periodic orbits are real!

Solution.
We define $f_n(x)$ with the recursive relation $f_1(x) = x$ and $f_{n+1}(x) = q(f_n(x))$ and are going to prove that $f_n(x) - x$ only has real solutions. For a periodic orbit we know there exists some $m > 1$ such that $a_m = a_1$, thus a_1 is a zero of $f_m(x) - x$.

We notice that $\deg f_n = 2^{n-1}$ and observe that $q(\cos(\theta)) = \cos(2\theta)$. With induction we can show that $f_n(\cos(\theta)) = \cos(2^{n-1}\theta)$.

For $k \in \{0, 1, \ldots, 2^{n-2}\}$ we define $a_k = \frac{2\pi k}{2^{n-1}+1}$. We can then calculate:

$$f_n(\cos(a_k)) = \cos(2^{n-1}a_k) = \cos\left(2\pi k \frac{2^{n-1}}{2^{n-1}+1}\right) = \cos\left(2\pi k\left(1 - \frac{1}{2^{n-1}+1}\right)\right) = \cos(a_k)$$

So $\cos(a_k)$ is a zero of $f_n(x) - x$. For $l \in \{1, \ldots, 2^{n-1}-1\}$ we can also look at $b_l = \frac{2\pi l}{2^n - 1}$ and see that $\cos(b_l)$ is a zero of $f_n(x) - x$. Since (a_k) and (b_l) are disjoint and all values are in $[0, \pi]$, we have found exactly 2^{n-1} different real zeros of $f_n(x) - x$. Since $\deg(f_n - x) = \deg(f_n) = 2^{n-1}$ we see all starting values of periodic orbits must be real, and thus all periodic orbits are real-valued.

12. Colouring the line with infinitely many colours

Prof. dr. D. (Dion) Gijswijt
Technische Universiteit Delft

We seek a surjective function $f : \mathbb{R} \to \mathbb{Z}$ such that for all $a, b, c \in \mathbb{R}$ the following holds:

$$a + c = 2b \implies \#\{f(a), f(b), f(c)\} < 3.$$

Does such a function exist?

Solution.
For a nonzero integer n, let $\mathrm{ord}_3(n)$ denote the largest integer k such that 3^k divides n. More generally, we define the 3-*adic order* $\mathrm{ord}_3 : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$ by

$$\mathrm{ord}_3\left(\frac{a}{b}\right) = \begin{cases} \infty & \text{if } a = 0, \\ \mathrm{ord}_3(a) - \mathrm{ord}(b) & \text{otherwise.} \end{cases}$$

We will show that a function f with the required properties exists. As an intermediate step, we define $g : \mathbb{R} \to \mathbb{Z} \cup \{\infty, -\infty\}$ by

$$g(x) = \begin{cases} -\infty & \text{if } x \notin \mathbb{Q} \\ \mathrm{ord}_3(x) & \text{if } x \in \mathbb{Q} \end{cases}$$

Let $a, b, c \in \mathbb{R}$ and suppose that $a + c = 2b$. We will show that $g(a)$, $g(b)$, and $g(c)$ are not distinct. If at most one of a, b, c is rational, this is clear. So we may assume that at least two of a, b, c are rational. Since $a + c = 2b$, this implies that $a, b, c \in \mathbb{Q}$.
For $a, c \in \mathbb{Q}$ we have

$$\mathrm{ord}_3(a + c) \geq \min(\mathrm{ord}_3(a), \mathrm{ord}_3(c))$$

with equality if $\mathrm{ord}_3(a) \neq \mathrm{ord}_3(c)$. So if $g(a) \neq g(c)$, then

$$g(b) = g(2b) = g(a + c) = \min(g(a), g(c)),$$

which implies that $g(b) = g(a)$ or $g(b) = g(c)$.
Let $h : \mathbb{Z} \cup \{-\infty, \infty\} \to \mathbb{Z}$ be a bijection. Then $f = h \circ g$ is of the desired form.